

PHY481 - Lecture 29

Chapter 9 of PS, Chapters 6,7 of Griffiths

A. Energy stored in inductors and in magnetic fields

An external voltage source must be used to set up a magnetic field in an inductor. The rate at which work is done by the external source is,

$$P = \frac{dW}{dt} = \mathcal{E}_{ext}I \quad (1)$$

We have shown that the voltage across an inductor is $-LdI/dt$, so we have,

$$\frac{dW}{dt} = -(-LI\frac{dI}{dt}) \quad (2)$$

The total energy stored in an inductor is the integral of the work so that,

$$U = \int \frac{dW}{dt}dt = \int_0^i LI\frac{dI}{dt}dt = \frac{1}{2}Li^2 \quad (3)$$

This energy is stored in the magnetic field of the inductor. By considering a solenoid we can find the energy density in the magnetic field, through

$$U = \frac{1}{2}Li^2 = \frac{1}{2}\mu_0n^2AlI^2 = \frac{1}{2\mu_0}B^2Al \quad (4)$$

Using $Al = volume$, the energy density in the magnetic field is then to be,

$$u = \frac{1}{2\mu_0}B^2 \quad (5)$$

We have introduced the concept of self-inductance which characterizes the ability of a wire configuration to store energy in the form of magnetic fields. This energy storage can be large and has been proposed as an alternative to batteries, provided superconducting wires are used in order to minimize the resistive losses. As you will see in the assignment very large amounts of energy are stored in the earth's magnetic field and in the magnetic fields of galaxies.

B. Mutual inductance

Self-inductance is defined through $N\phi = Li = \lambda$, where λ is the flux linkage. Self inductance is a measure of the ability of a conducting system to react to a current flow in the system. However, from consideration of Faraday's law, it is clear that the magnetic field

of one current carrying wire configuration may influence a second one that is not physically connected to it. This coupling is described by the mutual inductance. We write,

$$N_2\phi_2 = M_{12}i_1 \quad \text{so that} \quad V_2 = -N_2\frac{d\phi_2}{dt} = -M_{12}\frac{di_1}{dt} \quad (6)$$

This coupling is the basis of antennas, transformers and a variety of other essential elements of technology.

Calculations of mutual inductance are similar to that of self-inductance. We consider two examples.

A rectangular loop near a wire carrying current i_1 , and $N_2 = N_1 = 1$

We take the loop to have its long sides, of length l_2 , parallel to a long straight wire carrying current i_1 . The side which is closest to the wire is distance a from the wire. The short sides have length b_1 , and the normal to the loop is in the $\hat{\phi}$ direction with respect to the axis of the wire. In that case, we need to find $\phi_2 = M_{12}i_1$, so we have a by now quite familiar calculation,

$$\phi_2 = \int_0^l dy \int_a^{a+b} \frac{\mu_0 i}{2\pi r} = \frac{\mu_0 i l}{2\pi} \ln((a+b)/a), \quad \text{so that} \quad M_{12} = \frac{\mu_0 l}{2\pi} \ln((a+b)/a) \quad (7)$$

Two solenoids - one inside the other

The two solenoids are characterized by cross-sectional areas A_1, A_2 turns N_1, N_2 and lengths l_1, l_2 , with the former having $A_1 > A_2$. We also take the lengths of both solenoids to be much larger than their radii, so the formula $B = ni\mu_0$ applies, with $n = N/l$.

If a current i_1 flows in the larger solenoid, the flux in the smaller solenoid is then $\phi_2 = N_1 i_1 A_2 \mu_0 / l_1$ and the so using,

$$N_2\phi_2 = M_{12}i_1 \quad \text{implies that} \quad M_{12} = N_1 N_2 \mu_0 A_2 / l_1 \quad (8)$$

Now lets consider the reverse situation where a current i_2 flows in the smaller solenoid. In that case there is no magnetic field outside the smaller solenoid, so the flux in the larger solenoid is, $\phi_1 = N_2 i_2 \mu_0 / l_2$, then using

$$N'_1\phi_1 = M_{12}i_2 \quad \text{implies that} \quad M_{21} = N'_1 N_2 \mu_0 A_2 / l_2 \quad (9)$$

In this expression we use A_2 because we assume that there is no field outside the smaller solenoid. N'_1 is the number of loops in length l_2 , so $N'_1 = l_2 n_1 = l_2 N_1 / l_1$. With this substitution it is evident that $M_{21} = M_{12}$, so the mutual inductance is symmetric. Now

suppose we want to find the relation between current and voltage in the first solenoid. We have,

$$V_1 = -L \frac{di_1}{dt} - M \frac{di_2}{dt} \quad (10)$$

That is, both the directly coupled loop and the remotely coupled loop oppose a change in flux. We will return to this later after we have studied magnetic materials, as it is most naturally defined in terms of transformers where magnetic materials are used to control the direction that magnetic flux takes.

C. Maxwell's equations in magnetic materials

Diamagnetic materials expel magnetic flux, while paramagnetic and ferromagnetic materials provide favorable pathways for magnetic flux. Strongly diamagnetic materials, such as type I superconductors, expel flux so strongly that they can levitate when placed on top of a magnetic pole. Strongly ferromagnetic materials provide such good pathways for flux that they are used in magnetic circuits such as in transformers where flux is directed from the primary coil to the secondary coil. Understanding of these responses starts with an understanding of magnetization and of currents that lead to magnetization.

Recall that in dielectric materials we defined the polarization, \vec{P} as the electric dipole moment per unit volume. In magnetic materials, we define magnetization, \vec{M} to be the magnetic moment per unit volume. As in the discussion of dielectric materials, we shall discuss the general formulation and then concentrate on linear isotropic magnetic materials where $\vec{M} = \chi_m \vec{H}$. Recall that linear dielectric materials obey $\vec{P} = \epsilon_0 \chi_e \vec{E}$.

Recall that bound charges produce polarization and that at surfaces the bound charge is related to the polarization through $\sigma_b = \hat{n} \cdot \vec{P}$, while the bulk bound charge density is $\rho_b = -\vec{\nabla} \cdot \vec{P}$. Since the sources of magnetic field in Maxwell's equations of magnetostatics are steady currents, it is evident that the bound currents can produce magnetization. The relations between bound currents and the magnetization are,

$$\vec{K}_b = \vec{M} \wedge \vec{n}; \quad \vec{j}_b = \vec{\nabla} \wedge \vec{M} \quad (11)$$

The reasoning behind these expression is similar to that used to understand bound charges. Basically if the magnetization is a constant, only surface currents are required to produce the magnetization, while if the magnetization varies in space, a bulk bound current density is needed.

With these definitions, Ampere's law becomes,

$$\vec{\nabla} \wedge \vec{B} = \mu_0(\vec{j}_f + \vec{j}_b) = \mu_0(\vec{j}_f + \vec{\nabla} \wedge \vec{M}) = \mu_0(\vec{\nabla} \wedge \vec{H} + \vec{\nabla} \wedge \vec{M}) \quad (12)$$

where \vec{H} is the magnetic intensity and obeys,

$$\vec{\nabla} \wedge \vec{H} = \vec{j}_f \quad \text{so that} \quad \oint \vec{H} \cdot d\vec{l} = i_f; \quad \text{and} \quad \vec{B} = \mu_0(\vec{H} + \vec{M}) \quad (13)$$

From this equation it is seen that Amperian contours can be used to relate i_f to the magnetic intensity (or Auxilliary field) \vec{H} . The magnetic intensity and magnetization have units Amp per meter (A/m) as can be seen from the definition of \vec{H} .

Two types of magnetic material problems are typically posed. (i) Given a magnetisation, find the magnetic field, e.g. inside a uniformly magnetized cylinder. (ii) Given a linear magnetic material find the magnetic field inside and outside the material, e.g. a linear magnetic sphere in the presence of a uniform applied magnetic field. Later, we will look again at solenoids and transformers to see the effect of using magnetic materials in these systems, particularly the effect of magnetic materials on the energy stored in an inductor.

A uniformly magnetized cylinder with length \gg radius

We take the magnetization to be uniform, $\vec{M} = M_0 \hat{k}$ and to be oriented along the axis of the cylinder. There are no free currents, so there is no gain in using the magnetic intensity. To find the magnetic field, note that a uniform magnetisation is like the uniform field inside a solenoid, so surface currents flowing around the cylinder provide a solution. The system is linear so if we find a solution it is unique. Alternatively we can find the surface currents through,

$$\vec{K}_b = \hat{M} \wedge \hat{r} = M_0 \hat{\phi}; \quad \text{so that} \quad \vec{B}^{int} = \mu_0 K_b \hat{k} \quad (14)$$

Where the magnetic field inside the cylinder is then found using an Amperian loop and assuming that the field outside the cylinder is zero. This result applies in central regions of the magnetized cylinder. If we now consider a point \vec{r} which is well removed from the cylinder, i.e. $r \gg l$, then we can treat the cylinder as a dipole with dipole moment $\vec{m}_{cyl} = Volume \times \vec{M}$, and then use the dipole formula,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{3(\vec{m}_{cyl} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} \quad (15)$$

A Uniformly magnetized sphere of radius R

Take the magnetization to be, $\vec{M} = M_0 \hat{k}$, so that the bound currents are now,

$$\vec{K}_b = \vec{M} \wedge \hat{r} = M_0 \sin\theta \hat{\phi} \quad (16)$$

Assume that the magnetic field inside is uniform and along the \hat{k} direction, and that outside it is like a dipole, so that,

$$\vec{B}^{int} = B_0 \hat{k}; \quad \text{and} \quad \vec{B}^{ext} = \frac{\mu_0}{4\pi} \frac{3(\vec{m}_{cyl} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} \quad (17)$$

The boundary conditions on the magnetic field at the sphere surface are, $B_t^{ext} - B_t^{int} = \mu_0 K_b$, $B_n^{ext} = B_n^{int}$. The latter equation gives,

$$B_0 \hat{k} \cdot \hat{r} = B_0 \cos\theta = \frac{\mu_0}{4\pi} \frac{2\vec{m}_{cyl} \cdot \hat{r}}{R^3} = \frac{\mu_0}{4\pi} \frac{2m_{cyl} \cos\theta}{R^3} \quad (18)$$

From this expression, we find that

$$B_0 = \frac{\mu_0}{4\pi} \frac{2M_0 4\pi R^3}{3R^3} = \frac{2\mu_0 M_0}{3} \quad (19)$$

and hence,

$$\vec{B} = \frac{2\mu_0 M_0}{3} \cos\theta \hat{k} \quad (20)$$

It is easy to show that the boundary condition on the tangential component is also satisfied, so we have the correct solution. Notice that this solution differs by a factor of two from the analogous solution for a uniformly polarization sphere.