PHY481 - Lecture 3

Sections 2.2-2.5 of Pollack and Stump (PS)

The Divergence

The divergence is the dot product of the gradient operator and a vector function, $\vec{F} = (F_x, F_y, F_z)$, so that

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{1}$$

We want to find a co-ordinate independent representation of the divergence, which we achieve by considering a small cube of dimension ϵ. Now consider an integral of the flux through this surface, that is,

$$\oint_{dS} \vec{F} \cdot \hat{n} dA = \sum_{i=1}^{3} [F_i(\vec{x} + \epsilon \hat{e}_i/2) - F_i(\vec{x} - \epsilon \hat{e}_i/2)]\epsilon^2 \tag{2}$$

where $(\hat{e}_1, \hat{e}_2, \hat{e}_3) = (\hat{i}, \hat{j}, \hat{k})$ are introduced to simplify the expression. Eq. (22) reduces to,

$$\oint_{dS} \vec{F} \cdot \hat{n} dA = \sum_{i=1}^{3} \frac{\partial F_i(\vec{x})}{\partial x_i} \epsilon^3 = (\vec{\nabla} \cdot \vec{F})\epsilon^3 \tag{3}$$

The co-ordinate independent representation of the divergence of a vector function is then,

$$\vec{\nabla} \cdot \vec{F} = lim_{V \to 0} \frac{1}{V} \oint_{dS} \vec{F} \cdot d\vec{A} \tag{4}$$

The divergence is then proportional to the flux of the function, $\vec{F}$ though the surface of the volume V.

The Laplacian

The Laplacian ∇^2 is a scalar operator found by taking the dot product of the gradient operator with itself, ie.,

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}; \quad \text{in cartesion co} - \text{ordinates} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{5}$$

The Curl

The curl of a vector function, $\vec{\nabla} \wedge \vec{F}$ is defined in the same way as the cross product

$$\vec{\nabla} \wedge \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix} = (\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}) \tag{6}$$

Using Levi-Civita and suffix notation, this is,

$$(\vec{\nabla} \wedge \vec{F})_i = \varepsilon_{ijk}\frac{\partial F_k}{\partial x_j} \tag{7}$$

To find a co-ordinate independent representation, consider a square loop placed in the x-y plane, with edge length ϵ. Consider a path integral around the loop, the circulation,

$$\oint_{loop} \vec{F} \cdot d\vec{l} = \epsilon F_i(\vec{x} - \epsilon\hat{e}_j/2) + \epsilon F_j(\vec{x} + \epsilon\hat{e}_i/2) - \epsilon F_i(\vec{x} + \epsilon\hat{e}_j/2) - \epsilon F_j(\vec{x} - \epsilon\hat{e}_i/2) \tag{8}$$

This reduces to,

$$\oint_{loop} \vec{F} \cdot d\vec{l} = (\frac{\partial F_j}{\partial x_i} - \frac{\partial F_i}{\partial x_j}) = (\vec{\nabla} \wedge \vec{F})_k \epsilon^2 \tag{9}$$

The general co-ordinate independent form of $\vec{\nabla} \wedge \vec{F}$ is then

$$\hat{n} \cdot (\vec{\nabla} \wedge \vec{F}) = lim_{A\to 0}\frac{1}{A}\oint_C \vec{F} \cdot d\vec{l} \tag{10}$$

F. Derivative operator identities

Table 2.2 of PS gives a list of indentities for derivative operators. We will look at a couple of interesting examples and in the homework you will need to use both these identities and the vector identities in Table 2.1 of the text.

Identity. $\vec{\nabla} \wedge (\vec{\nabla} f) = 0$. *Proof:* $(\vec{\nabla} \wedge \vec{\nabla} f)_i = \varepsilon_{ijk}\frac{\partial^2 f}{\partial x_j \partial x_k}$. Because there is a sum over j, k pairs of terms appear with opposite signs, due to the assymmetry of the Levi-Civita tensor. Summing these pairs of terms gives zero.

Identity. $\vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{F}) = 0$. *Proof:* Using Levi-Civita notation, the LHS is $\varepsilon_{ijk}\frac{\partial^2 F_k}{\partial x_i \partial xj}$. Again the terms on the RHS can be collected into pairs with opposite signs due to the antisymmetric nature of the Levi-Civita tensor. The identity is then proved.

G. Summary of key geometrical concepts

This has been a busy lecture with many mathematical details. If you have not seen these mathematical tools before, it will take some time for you to become familiar with them. However the most important lessons to take from these tools is the key concepts that enable insight into the physics behind the math. Here are four key concepts.

1. The cross product $\vec{A} \wedge \vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$.

2. The definition of the gradient through $df = d\vec{x} \cdot \vec{\nabla} f$ demonstrates that $\vec{\nabla} f$ is perpendiclar to equipotentials of the function f.

3. The divergence of vector function $\vec{\nabla} \cdot \vec{F}(\vec{x})$ gives the flux emanating through an infinitesimal volume encompassing $\vec{x}$.

4. The quantity $\hat{n} \cdot \vec{\nabla} \wedge \vec{F}$ gives the circulation of $\vec{F}$ around a loop in the plane defined by $\hat{n}$ and around the point $\vec{x}$.

Integral Theorems

Two integral theorems follow from the definitions of the divergence and the curl. The integral theorem which follows from the definition of the divergence (Eq. 24) is **Gauss's theorem** which states that the net flux of a vector function $\vec{F}$ through a closed surface S is equal to an integral of the divergence of $\vec{F}$ over the volume enclosed, ie.,

$$\oint_S \vec{F} \cdot \hat{n} dA = \int_V \vec{\nabla} \cdot \vec{F} dV \tag{11}$$

Notice that Eq. (24) is this equation in the limit where the enclosed volume goes to zero. The extension to any closed surface is understood by dividing the large volume into boxes and noting that the only flux that remains is the flux through the outer surfaces. The interior boxes cancel and the normal to interior planes takes on the two possible directions, canceling in the total integral on the RHS. We shall use Gauss's law to demonstrate that the differential form of Gauss's law in electrostatics and in magnetostatics follows from the integral forms of these laws.

Stokes law follows from the definition of the curl (Eq. 30) and is again an extension of that relation to finite domains,

$$\oint_C \vec{F} \cdot d\vec{l} = \int_S (\vec{\nabla} \wedge \vec{F}) \cdot d\vec{A} \tag{12}$$

The extension again follows from breaking up the finite domain into small squares and noting that all interior loops segments have zero net contribution to the contour integral, due to the opposite sense of the circulation in that wire from the two squares which are adjacent to the wire. We shall use Stokes theorem to relate the integral form of Faraday's law and Ampere's law to their differential forms.