PHY481 - Lecture 4

Sections 2.4,2.5 of Pollack and Stump (PS)

A. Curvilinear co-ordinates: Scale factors h_1, h_2, h_3

In general a set of curvilinear co-ordinates can be orthogonal or non-orthogonal. We focus on the orthogonal case, which includes cartesion, cylindrical and spherical co-ordinates. We denote the unit vectors to be $\hat{e}_1, \hat{e}_2, \hat{e}_3$, and that a position vector $\vec{x}$ is written as,

$$\vec{x} = (u_1, u_2, u_2) = u_1\hat{e}_1 + u_2\hat{e}_2 + u_3\hat{e}_3. \tag{1}$$

For the Cartesion, Cylindrical, and Spherical Polar cases, we have $(u_1, u_2, u_2) = (x, y, z), (r, 0, z), (r, 0, 0)$ respectively.

Now imagine displacing the co-ordinates by a small amount du_1, du_2, du_3, this leads to a change in the vector $\vec{x}$ by an amount $d\vec{s}$. In general we can write

$$d\vec{s} = u_1h_1\hat{e}_1 + u_2h_2\hat{e}_2 + u_3h_3\hat{e}_3 \tag{2}$$

where h_1, h_2, h_3 are the scale factors. They are central to deriving the relations that we need. Lets derive them for the three cases of interest (done in class). The results are,

Cartesian: $h_x = 1, h_y = 1, h_z = 1$

Cylindrical: $h_r = 1, h_\phi = r, h_z = 1$

Spherical Polar: $h_r = 1, h_\theta = r, h_\phi = rsin\theta$

B. Unit Vector Transformations

In lecture 2 we noted the transformations between Cartesian and cylindrical co-ordinates,

$$x = rcos\phi; \quad y = rsin\phi; \quad x^2 + y^2 = r^2 \tag{3}$$

and that between Cartesian and polar co-ordinates,

$$x = rcos\phi sin\theta; \quad y = rsin\phi sin\theta; \quad z = rcos\theta; x^2 + y^2 + z^2 = r^2 \tag{4}$$

Relations between unit vectors in the three co-ordinate systems are also useful. First consider cylindrical co-ordinates (where we use either θ or ϕ for the angle and r for the in plane radius). It is easy to see that,

$$\hat{r} = cos\phi\hat{i} + sin\phi\hat{j}; \quad \hat{\phi} = -sin\phi\hat{i} + cos\phi\hat{j} \tag{5}$$

and

$$\hat{i} = cos\phi\hat{r} - sin\phi\hat{\phi}; \quad \hat{j} = sin\phi\hat{r} + cos\phi\hat{\phi} \tag{6}$$

The transformations in the case of polar co-ordinates are,

$$\hat{r} = sin\theta cos\phi\hat{i} + sin\theta sin\phi\hat{j} + cos\theta\hat{k};$$

$$\hat{\theta} = cos\theta cos\phi\hat{i} + cos\theta sin\phi\hat{j} - sin\theta\hat{k};$$

$$\hat{\phi} = -sin\phi\hat{i} + cos\phi\hat{j} \tag{7}$$

and

$$\hat{i} = sin\theta cos\phi\hat{r} + cos\theta cos\phi\hat{\theta} - sin\phi\hat{\phi};$$

$$\hat{j} = sin\theta sin\phi\hat{r} + cos\theta sin\phi\hat{\theta} + cos\phi\hat{\phi};$$

$$\hat{k} = cos\theta\hat{r} - sin\theta\hat{\theta} \tag{8}$$

C. Volume element, Grad, Div, Curl and Laplacian in curvilinear co-ordinates

We write down the expressions for these quantities in terms of the scale factors, h_i, and explicit expressions are found by A *volume element* dV at position $\vec{x}$ is given by,

$$dV = h_1 h_2 h_3 du_1 du_2 hu_3 \tag{9}$$

1. The gradient is found by using two forms for the expansion of a scalar function

$$df = \frac{\partial f}{\partial u_1}du_1 + \frac{\partial f}{\partial u_2}du_2 + \frac{\partial f}{\partial u_3}du_3 \tag{10}$$

and

$$df = d\vec{s}\cdot\vec{\nabla}f = ds_1(\vec{\nabla}f)_1 + ds_2(\vec{\nabla}f)_2 + ds_3(\vec{\nabla}f)_3 \tag{11}$$

Using Eq. (), we also have

$$ds_1 = h_1 du_1; \quad ds_2 = h_2 du_2 \quad ds_3 = h_3 du_3 \tag{12}$$

Combining Eq. (), yields,

$$\vec{\nabla} = (\frac{1}{h_1}\frac{\partial}{\partial u_1}, \frac{1}{h_2}\frac{\partial}{\partial u_2}, \frac{1}{h_3}\frac{\partial}{\partial u_3}) \tag{13}$$

2. The divergence is found by using the same procedure that we used in Eq. (), but now using curvilinear co-ordinates.

$$\vec{\nabla}\cdot\vec{F}dV = [F_1 h_2 h_3|_{u_1+du_1} - F_1 h_2 h_3|_{u_1}]du_2 du_3 +$$

$$[F_2h_1h_3|_{u_2+du_2} - F_2h_1h_3|_{u_2}]du_1du_3 + [F_3h_1h_2|_{u_3+du_3} - F_3h_1h_2|_{u_3}]du_1du_2 \tag{14}$$

From this expression and using $dV = h_1h_2h_3du_1du_2du_3$ we find,

$$\vec{\nabla} \cdot \vec{F} = \frac{1}{h_1h_2h_3}[\frac{\partial}{\partial u_1}(F_1h_2h_3) + \frac{\partial}{\partial u_2}(F_2h_1h_3) + \frac{\partial}{\partial u_3}(F_3h_1h_2)] \tag{15}$$

3. The Laplacian of a scalar function f is $\vec{\nabla} \cdot (\vec{\nabla} f)$, which reduces to,

$$\nabla^2 f = \frac{1}{h_1h_2h_3}\sum_{(ijk)} \frac{\partial}{\partial u_i}(\frac{h_jh_k}{h_i}\frac{\partial f}{\partial u_i}) \tag{16}$$

4. The Curl is developed in a similar way yielding,

$$\vec{\nabla} \wedge \vec{F} = \begin{vmatrix} \hat{i}/h_2h_3 & \hat{j}/h_3h_1 & \hat{k}/h_1h_2 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} \tag{17}$$

These expressions are used, along with the expressions for the scale factors h_i to write down expressions for Grad, Div, Curl and Laplacian in cylindrical and in polar co-ordinates. We shall use these expressions repeatedly later in the course.

D. Helmholtz theorem. Any vector field $\vec{F}$ with the properties

$$lim_{r\to\infty}\vec{\nabla} \cdot \vec{F} \to 0, \quad lim_{r\to\infty}\vec{\nabla} \wedge \vec{F} \to 0,$$

may be broken up into a divergence free part and an irrotational (curl free) part, so that,

$$\vec{F} = -\vec{\nabla}\Phi + \vec{\nabla} \wedge \vec{A} \tag{18}$$

where the scalar potential Φ and the vector potential $\vec{A}$ are related to the vector function $\vec{F}$ through.

$$\Phi = -\int_V \frac{\vec{\nabla} \cdot \vec{F}}{4\pi|\vec{r'} - \vec{r}|}d\vec{r'} \tag{19}$$

$$\vec{A} = -\int_V \frac{\vec{\nabla} \wedge \vec{F}}{4\pi|\vec{r'} - \vec{r}|}d\vec{r'} \tag{20}$$

As we shall see later, these expression are derived in physics by superposition of the electrostatic potential (topic expression) and vector potential (lower expression).

A consequence of Helmholtz's theorem is that if we know the div and curl of a vector function, then the function is specified. This is important in electrostatics where we have the two equations

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0; \quad \vec{\nabla} \wedge \vec{E} = 0 \tag{21}$$

and

$$\vec{\nabla} \cdot \vec{B} = 0; \quad \vec{\nabla} \wedge \vec{B} = \mu_0 j \tag{22}$$

Now recall the two identities that are valid for any scalar function f and vector function $\vec{F}$

$$\vec{\nabla} \wedge (\vec{\nabla} f) = 0; \quad \vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{F}) = 0 \tag{23}$$

From these identities it is easy to see that $\vec{E} = -\vec{\nabla}\psi$ and $\vec{B} = \vec{\nabla} \wedge \vec{A}$, where ψ is the electrostatic potential and $\vec{A}$ is the vector potential.

Examples

1. Consider $f = x$, then $\vec{\nabla} = \hat{\imath}$. Equipotentials are at $x = \textit{constant}$, so $\vec{\nabla}$ is perpendicular to equipotentials of f.

2. Consider the two functions $\vec{F} = x\hat{\imath}$ and $\vec{F} = x\hat{\jmath}$. Show that the first function is irrotational (curl is zero) while the second is solenoidal (divergence is zero).