PHY481 - Lecture 5

Sections 3.1-3.5 of PS

All of electrostatics follows from Coulomb's law + superposition

A. Coulomb's law - Force between two charges

The starting point in electrostatics is Coulomb's law, which gives the force between two stationary charges,

$$\vec{F} = k\frac{Qq}{r^2}\hat{r} = k\frac{Qq}{r^3}\vec{r} \tag{1}$$

- Q, q are stationary charges. Their units are coulombs (C)
- $\hat{r}$ is a unit vector along the line between the two charges.
- $\vec{r}$ is the vector distance between the two charges.
- $k = 9 \times 10^9 Nm^2/C^2 = 9 \times 10^9 kgm^3/C^2s^2$.
- $k = 1/4\pi\epsilon_0$. ϵ_0 is the permittivity of free space.

B. Force between many charges - superposition

Force on a charge q due to many other charges, $Q_1, Q_2, ...Q_n$ is just the sum of the forces due to each of these charges, ie.

$$\vec{F}_{tot} = \sum_{i=1}^{n} k\frac{Q_i q}{r_i^2}\hat{r}_i \tag{2}$$

This looks simple, but of course it is a vector sum, so the math can get messy. r_i is the distance between charge i and the charge q. $\hat{r}_i$ is a unit vector along the vector which goes from charge q to charge Q_i.

The principle of superposition also applies when there is a continuous distribution of charge. For example charge distributions on rods, discs, spheres etc. However when treating these distibutions, the sum in Eq. (5) becomes an integral. In treating these problems, we define a small element of charge dQ. This is the amount of charge in a small part of the continuous charge distribution. We shall consider three cases:

- Lines: Then $dQ = \lambda dx$, where λ is the linear charge density.
- Surfaces: Then $dQ = \sigma dA$, where σ is the surface charge density.
- Volumes: Then $dQ = \rho dV$, where ρ is the volume charge density.

Example - uniform ring of charge

Consider a thin perfectly circular ring centered at the origin, with radius R, lying in the x-y plane and being uniformly charged with linear charge density λ. Find the force on a charge q placed at position, $\vec{r} = (0, 0, z)$.

Solution

By symmetry, the x and y components of the force on the charge are zero. We are thus left with the task of calculating the z-component. The distance, r, from the charge to any point on the ring is given by,

$$r^2 = R^2 + z^2 \tag{3}$$

If we define θ to be the angle that the vector $\vec{r}$ makes with the z-axis, then,

$$cos(\theta) = z/r \tag{4}$$

The total force on the charge is then,

$$F_z = \int_0^{2\pi} \frac{kqdQ}{r^2} cos(\theta) = \frac{kq}{R^2 + z^2} \frac{z}{r} \int_0^{2\pi} \lambda R d\phi = \frac{kqQz}{(R^2 + z^2)^{3/2}} \tag{5}$$

where we used $Q = 2\pi R\lambda$

C. The electric field

Definition: Electric field = force on a unit charge

The electric field at a point $\vec{r}$ due to a charge distribution is defined in terms of the force on a positive test charge, q, placed at position $\vec{r}$. The precise definition is,

$$\vec{E}(\vec{r}) = \mathrm{Lim}_{q\to 0} \frac{\vec{F}_q(\vec{r})}{q} \tag{6}$$

We can think of the electric field as the force per unit positive charge at position $\vec{r}$. Notice that once we have found $\vec{E}(\vec{r})$, we can find the force on any charge, q, by using $\vec{F}(\vec{r}) = q\vec{E}(\vec{r})$. Since the Electric field is basically a force, superposition applies to the electric field. All of the properties of the electric field can be derived from Eq. () and superposition, as we now show.

First, from the definition of the electric field, we write Coulomb's law as,

$$\vec{E}(\vec{r}) = k\frac{Q}{r^2}\hat{r} \tag{7}$$

which is the electric field of a point charge. In a similar way, the electric field due to a distribution of discrete charges is given by,

$$\vec{E}(\vec{r}) = \sum_i k\frac{Q_i}{r_i^2}\hat{r}_i \tag{8}$$

Below we show how the integral and differential forms of Gauss's law follow from this expression. An extremely useful concept in developing these results is the concept of an electric field line. An electric field line is a series of vectors where at each point the vector points in the direction of the force on a unit charge at that point and it has a length equal to the magnitude of the force. ie. we plot the vector function $\vec{E}$. The proporties of electric field lines constructed in this way are as follows.

++++++++++++++++++++++++++++++++++++++

Example- discrete charge distribution

Consider three charges located (and fixed) at (x,y) positions $\vec{r_1} = (0,0)$, $\vec{r_2} = (2,0)$, $\vec{r_3} = (0,2)$ (in meters) and having charges $q_1 = q_2 = 2nC$, $q_3 = -3nC$. Find the force acting on charge q_3 due to the other two charges. What is the electric field at position $\vec{r}_3$ due to charges q_1 and q_2?

Solution

We need to treat the force as a vector. In this problem the best way to treat the force is to resolve the Coulomb force between each pair of charges into its x-components and it y-components. The charges form an isosceles right-angle triangle, with charge q_1 lying at the right-angle corner. The other angles in the triangle are 45^o. The x-component of the force on charge q_3 is given by,

$$f_x = k\frac{q_2 q_3}{r_{23}^2}cos(45) + 0 \tag{9}$$

The y-component of the force on charge q_3 is given by,

$$f_y = k\frac{q_2 q_3}{r_{23}^2}sin(45) + k\frac{q_1 q_3}{r_{13}^2} \tag{10}$$

The electric field at position $\vec{r}_3$ due to charges q_1 and q_2 is,

$$E_x = f_x/q_3 = k\frac{q_2}{r_{23}^2}cos(45); \quad E_y = f_y/q_3 = k\frac{q_2}{r_{23}^2}sin(45) + k\frac{q_1}{r_{13}^2} \tag{11}$$

Example - uniform ring of charge

Find the electric field for the thin charge ring discussed above.

Solution

$$E_x = 0; \quad E_y = 0; \quad E_z = F_z/q = \frac{kQz}{(R^2 + z^2)^{3/2}} \tag{12}$$

++

D. Electric field lines - Faraday's ideas

1. At each point along an electric field line, the force on a positive test charge is in a direction tangent to the field line at that point.

2. The density of lines at any point in space is proportional to the magnitude of the electric field at that point.

3. Electric field lines begin and/or end at charges, or they continue off to infinity. i.e. they do not begin or end in free space.

4. Electric field lines do not cross.

Very important special case: conductors

1. If there is no current flowing, then the electric field is zero, $\vec{E} = 0$, inside a conductor.

2. If there is no current flowing, then at the surface of a conductor, the electric field is normal to the surface of the conductor.

E. Gauss's Law

The integral form of Gauss's law in free space is,

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{q_{encl}}{\epsilon_0} \tag{13}$$

where q_{encl} is the total charge inside the closed surface S. This law follows from Coulomb's law and superposition in combination with the properties of electric field lines. The proof of Gauss's law in general follows from the following statements.

Property 1. (i) For a charge q with a spherical shell at radius r it is easy to prove that Gauss's law is correct. (ii) For a charge that is outside of a spherical shell it is also easy to prove that the total flux through the shell is zero. This is proven by noting all flux lines originate or terminate at a charge, or go to infinity. Therefore a flux line originating from a charge outside the spherical shell and which enters the spherical shell must also leave the spherical shell. The net flux through the surface of the shell due to that flux line is then zero.

Property 2. Flux lines are like a conserved fluid flow so that any surface drawn around a charge must have the same total flux through it. This gives us a great deal of freedom in drawing the surfaces through which flux flows. For a single charge clearly a spherical surface is most convenient.

Property 3. If we have a distribution of charge inside a Gaussian surface, we can break the charge up into small pieces and treat each piece with a spherical surface, and the total flux through a surface surrounding all of the charges is the same as the sum of the flux due to each little piece of charge through a spherical surface surrounding that charge. This property is due to the superposition property that is correct in electrostatics.

The q_{encl} on right hand side of Eq. (13) is the total charge in the volume V enclosed by the closed surface S, which may be written in either discrete or continuous forms,

$$q_{encl} = \sum_{i \varepsilon V} q_i = \int_V \rho(\vec{r}) d\vec{r} \tag{14}$$

where $\rho(\vec{r})$ is the charge density.

The differential form of Gauss's law follows from using the last expression on the RHS of this equation and applying the divergence theorem to the LHS of Eq. (13) to find,

$$\int_V \vec{\nabla} \cdot \vec{E} dV = \frac{1}{\epsilon_0} \int_V \rho dV \quad \text{which implies} \quad \vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \tag{15}$$

Note that we can also write an integral form for the electric field due to a continuous charge distribution through direct application of the superposition principle,

$$\vec{E}(\vec{r}) = \int_V \frac{k\rho(\vec{r'})(\vec{r} - \vec{r'}) d\vec{r'}}{|\vec{r} - \vec{r'}|^3}. \tag{16}$$

Finally it is possible to demonstrate Eq. (15) from Eq. (16) and also by taking the curl of Eq. (16) that

$$\vec{\nabla} \wedge \vec{E} = 0. \tag{17}$$

Once the electric field has been determined, we can find the force on an arbitrary charge through $\vec{F} = Q\vec{E}$.