PHY481 - Lecture 6

Sections 3.4-3.6 of PS

A. Finding the electric field

We have three methods for finding the electric field in electrostatics.

(i) Superposition,

$$\vec{E}(\vec{r}) = \sum_i \frac{kq_i(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3} \quad \text{or} \quad \oint_S \frac{k\rho(\vec{r'})(\vec{r}-\vec{r'})d\vec{r'}}{|\vec{r}-\vec{r'}|^3}. \tag{1}$$

Slightly different continuous forms are used for line or surface charge cases.

(ii) Gauss's law in integral form,

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{q_{encl}}{\epsilon_0} \tag{2}$$

(iii) Gauss's law in differential form,

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \quad \text{and} \quad \vec{\nabla}\wedge\vec{E} = 0 \tag{3}$$

Chapter 5 of PS is devoted to the differential approach. Here we first concentrate on the first two approaches. Remember that in all electrostatics problems superposition can be used so if you can break the problem or structure of interest into a sum of structures whose solutions you know, then the solution is often straightforward to write down.

(i) Infinite sheet of charge - Charge density $\sigma(C/m^2)$

Gauss's law - Draw a rectangular box around the sheet. Note that by symmetry the electric field lines are perpindicular to the sheet. We then have,

$$\oint \vec{E}\cdot d\vec{A} = 2EA = \sigma A/\epsilon_0, \quad \text{or} \quad E = \sigma/2\epsilon_0. \tag{4}$$

(ii) Just outside the surface of a conductor - Charge density $\sigma(C/m^2)$

Use an infinitesimal rectangle and note that the electric field inside the conductor is zero, therefore,

$$\oint \vec{E}\cdot d\vec{A} = EdA = \sigma dA/\epsilon_0, \quad \text{or} \quad E = \sigma/\epsilon_0. \tag{5}$$

Note two things. (i) The electric field at the surface is twice that of a charge sheet with the same charge density. (ii) If an electric field is applied to a conductor, the charges move to screen out the applied field - the surfaces become charged leading to an *induced* dipole.

This is used to trap small particles using focused laser beams or laser tweezers. As we shall discuss later many molecules have permanent dipoles as well. The size of the induced dipole on a molecule depends on the *polarizeability* of the electron cloud of the molecule (metals are highly polarizeable)

(iii) Infinite sheet of uniform charge density σ using integration

Place the sheet in the x-y plane and note by symmetry that only E_z is finite. There is no dependence on the x-y co-orindates as there is translational symmetry in the x-y plane. To carry out the integral we use cylindrical co-ordinates and use the result for a ring of charge (Lecture 5) $E_z = kQz/(r^2+z^2)^{3/2}$,

$$E_z = \int_0^\infty \frac{kdQ(r)z}{(r^2+z^2)^{3/2}} \quad \text{with} \quad dQ(r) = \sigma(2\pi r dr), \tag{6}$$

so we find,

$$E_z = \int_0^\infty \frac{\sigma 2\pi r dr z}{4\pi\epsilon_0(r^2+z^2)^{3/2}} = \frac{-\sigma z}{2\epsilon_0(r^2+z^2)^{1/2}}|_0^\infty = \frac{\sigma}{2\epsilon_0} \tag{7}$$

The simplest charged capacitors consist of two sheets of metal with equal and opposite charges on the surfaces of the sheets. What is the electric field inside and outside this type of capacitor?

(iv) Infinite cylindrical shell of radius R

Assume the rod has radius R with its axis along the z-axis, and we use cylindrical co-ordinates. To apply Gauss's law, we note that by symmetry the electric field is in $\hat{r}$ direction. Consider two closed cylindrical Gaussian surfaces, S, of radius $r < R$ and $r > R$ respectively. In both cases the length of the cylinder is L. By symmetry there is no flux through the top and bottom of the cylinder. Moreover if $r < R$ there is no enclosed charge, so the electric field is zero. For the case $r > R$, we have,

$$\int_S \vec{E} \cdot d\vec{A} = 2\pi r L E(r) = 2\pi R L \sigma/\epsilon_0 \tag{8}$$

so that

$$\vec{E} = \frac{\sigma}{\epsilon_0}\frac{R}{r} = \frac{\lambda}{2\pi\epsilon_0 r} \tag{9}$$

where $\lambda = 2\pi R\sigma$ is the charge per unit length of the cylinder.

Capacitors are sometimes made using concentric cylinders which hold equal and opposite charge. The electric field is again found by adding the fields due to the two cylinders.

(v) Infinite rod by integration

To find this result using direct integration is somewhat involved so lets do the calculation in the limit $R \to 0$, which gives a rod with charge density λ. The result from this calculation should be the same as that above for the interesting case $r > R$. Then, using the fact that only the $\hat{r}$ component is finite,

$$E(r) = \int kdQ \frac{cos\theta}{(r^2 + z^2)} = 2\int_0^\infty k\lambda dz \frac{r}{(r^2 + z^2)^{3/2}} \tag{10}$$

Now use the integral result $\int dz/(r^2 + z^2)^{3/2} = z/(r^2(r^2 + z^2)^{1/2})$ to find

$$\vec{E} = 2k\lambda \frac{zr}{(r^2(r^2 + z^2)^{1/2}}|_0^\infty \to \frac{2k\lambda}{r} \tag{11}$$

Note that the upper limit yields the result and the lower limit is zero. This is valid as when $z \to \infty$, $z/(z^2 + r^2)^{1/2} \to 1$. (see Problem 3.8)