Midterm II - October 31st 2008

Maxwell's equations in integral and differential form are given by,

$$\oint \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0}; \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \quad \text{(Gauss's law)} \tag{1}$$

$$\oint \vec{B}\cdot d\vec{A} = 0; \quad \vec{\nabla}\cdot\vec{B} = 0 \quad \text{(no official name)} \tag{2}$$

$$\oint \vec{E}\cdot d\vec{l} = -\frac{\partial\phi_B}{\partial t}; \quad \vec{\nabla}\wedge\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{(Faraday's law)} \tag{3}$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 i + \mu_0\epsilon_0\frac{\partial\phi_E}{\partial t}; \quad \vec{\nabla}\wedge\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \quad \text{(Ampere − Maxwell law)} \tag{4}$$

In dielectric materials we introduce the polarization and the displacement field, with $\epsilon_0\vec{E} = \vec{D} - \vec{P}$, and we break the charge appearing in Maxwell's equations into two parts $\rho = \rho_f + \rho_b$ and $q = q_f + q_b$. The bound charge is related to the polarization through $\sigma_b = \hat{n}\cdot\vec{P}$, $\rho_b = -\vec{\nabla}\cdot\vec{P}$. The definition of the displacement field is,

$$\int \vec{D}\cdot d\vec{A} = q_f; \quad \vec{\nabla}\cdot\vec{D} = \rho_f \tag{5}$$

Finally in linear isotropic dielectrics we have the constitutive law $\vec{D} = \epsilon\vec{E} = \kappa\epsilon_0\vec{E}$.

An integral that might be useful,

$$\int_0^{2\pi} sin(n\phi)sin(n'\phi)d\phi = \pi\delta_{nn'} \tag{6}$$

Problem 1

a) Derive the differential form of Eq. (3) from the integral form of Eq. (3). Show how this, along with Eq. (1), leads to Poisson's Equation in electrostatics.

Solution. Using Stokes Law and the definition of magnetic flux in the integral form of (3), i.e.,

$$\oint \vec{E} \cdot d\vec{l} = \int \vec{\nabla} \wedge \vec{E} \cdot d\vec{A}; \quad \phi_B = \int \vec{B} \cdot d\vec{A} \tag{7}$$

leads to the differential form of Faraday's law. The relation $\vec{\nabla} \cdot \vec{E} = 0$ implies that $\vec{E} = -\vec{\nabla} V$ (actually the minus sign is by convention). Subsitution of this into $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$, yields Poisson's equation.

b) A metal sphere of radius R is placed in a constant electric field $\vec{E}_0 = E_0 \hat{k}$. (i) Find the electric field outside the sphere $\vec{E}^{ext}(r, \theta, \phi)$. (ii) Find the charge density at the surface of the sphere.

Solution. Assuming that the sphere is grounded, the solutions for the electrostatic potential are given by,

$$V^{int} = 0; \qquad V^{ext} = -E_0 r cos(\theta) + \frac{AR^3}{r^2} cos(\theta) \tag{8}$$

The tangential electric field must be zero at the surface of the sphere, so that,

$$E_t^{ext}(r = R) = \frac{-1}{R} \frac{\partial V^{ext}}{\partial \theta}|_R = -E_0 sin(\theta) + A sin(\theta) = 0 \text{ so } \quad A = E_0 \tag{9}$$

The electric field outside the sphere is then,

$$E_r = -\frac{\partial V^{ext}}{\partial r} = E_0 cos\theta(1 + 2\frac{R^3}{r^3}); \qquad E_\theta = \frac{-1}{r} \frac{\partial V^{ext}}{d\theta} = E_0 sin\theta(-1 + \frac{R^3}{r^3}) \tag{10}$$

The charge density at the surface of the conducting sphere is given by,

$$\sigma = \epsilon_0 E_n(r = R) = \epsilon_0 \frac{-\partial V^{ext}}{\partial r}|R = \epsilon_0(E_0 cos(\theta) + 2E_0 cos(\theta)) = 3\epsilon_0 E_0 cos(\theta) \tag{11}$$

Problem 2

a) From the integral forms of Eq. (1) and Eq. (3), derive the boundary conditions on the normal and tangential electric fields at the surface of a dielectric material. Using Eq. (5)

derive the boundary condition on the displacement field at the surface.

Solution. To find the relation for the normal component of the displacement field at the surface we use the integral form of the electrostatic equations. Then, make a cubic Gaussian box about the surface and use Eq. (5), from which we find,

$$D_n^{ext} - D_n^{int} = \sigma_f. \tag{12}$$

In a similar way we can use the same Gaussian box for the electric field with Eq. (1) gives,

$$E_n^{ext} - E_n^{int} = \frac{\sigma_f + \sigma_b}{\epsilon_0}. \tag{13}$$

To find the relation for the tangential field, consider Faraday's law, with the RHS zero and form a loop about the surface, which gives,

$$E_t^{ext} - E_t^{int} = 0 \tag{14}$$

b) Find the electrical potentials V^{ext} and V^{int} due to a point charge above a dielectric half-space. The point charge, q, is at position d on the z-axis, and the dielectric half-space, with relative dielectric constant κ, lies in the region $z < 0$. Find the force acting on the point charge. Make sure that as $\kappa \to 1$, your force goes to zero.

Solution. The image charge method extends quite straightforwardly to a a point charge q at position d above a uniform dielectric in the lower half-space ($z < 0$). We assume that the electrostatic potential in the upper half space $z > 0$ is given by,

$$V_{above} = \frac{kq}{(x^2 + y^2 + (z-d)^2)^{1/2}} + \frac{kq'}{(x^2 + y^2 + (z+d)^2)^{1/2}} \tag{15}$$

The potential in the lower half-space is given by,

$$V_{below} = \frac{kq''}{(x^2 + y^2 + (z-d)^2)^{1/2}} \tag{16}$$

Continuity of the potential at $z = 0$ and the requirement that D_n be continuous, also at $z = 0$, lead to,

$$q'' = q + q'; \quad (q - q')d = -\kappa q'' d \tag{17}$$

Solving gives,

$$q' = q\frac{1-\kappa}{1+\kappa}; \quad q'' = q\frac{2}{1+\kappa} \tag{18}$$

The electric field inside the dielectric ($z < 0$)is thus like that of a point charge at the same position as the original charge, but with a reduced magnitude, q''.

The force acting on the image charge is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{d^2} = \frac{-q^2}{4\pi\epsilon_0(4d^2)}\frac{\kappa-1}{\kappa+1} \tag{19}$$

Problem 3

Consider a thin cylindrical shell of radius R, centered at the origin with its axis along the z-axis, that has a fixed potential of V_0 on its lower ($y < 0$)surface and that is grounded on its upper surface $y > 0$. Find the electrostatic potential in the interior region $r < R$, and in the exterior region $r > R$. What is the potential at the origin?

Solution. The solution $V(r, \phi)$ can be written as the sum of a constant $V_0/2$ plus the solution to a problem with boundary condition $V_0/2$ on the bottom and $-V_0/2$ on the top. The latter problem has solution

$$V^{ext} = \sum_{n=1}^{\infty} B_n(\frac{R}{r})^n sin(n\phi); \quad V^{int} = \sum_{n=1}^{\infty} A_n(\frac{r}{R})^n sin(n\phi) \tag{20}$$

Setting $V_{ext}(R) = V_{int}(R)$ shows that $A_n = B_n$. We choose the sin function as the boundary conditions are odd in ϕ. Fourier analysis gives,

$$\int_0^{2\pi} V_{boundary} sin(n'\phi) = \int_0^{\pi} \frac{-V_0}{2} sin(n'\phi)d\phi + \int_{\pi}^{2\pi} \frac{V_0}{2} sin(n'\phi)d\phi = \pi A_n \delta_{nn'} \tag{21}$$

From this we find, $A_n = \frac{-4V_0}{\pi n}$ for n odd, and zero for n even. The potential at the origin is $V_0/2$