A. Magnetostatics

Mathematical essentials: Stokes theorem, Divergence theorem, and identities:

$$\vec{\nabla} \wedge \vec{\nabla} f = 0; \quad \vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{F}) = 0; \quad \vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{F}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \vec{\nabla}^2 \vec{F} \tag{1}$$

(i) Sources of magnetic fields: Currents, intrinsic magnetic moments of elementary particles.

(ii) Calculating magnetic fields produced by currents:

- Ampere's law $\oint \vec{B} \cdot d\vec{l} = \mu_0 i$. Typical problems: Current sheet, wires/concentric cylinders, solenoid.

- Biot-Savart Law $d\vec{B} = \frac{\mu_0 i}{4\pi} \frac{d\vec{l} \wedge \vec{R}}{R^3}$. Typical problems: circular ring, square loop. Prove that the magnetic moment of loops is $\vec{m} = i\vec{a}$, where $\vec{a}$ is the area of the loop.

(iii) Magnetic field of a dipole

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} \tag{2}$$

The usual dipole formula apply: $\vec{\tau} = \vec{m} \wedge \vec{B}$; $U = -\vec{m} \cdot \vec{B}$.

(iv) Lorentz force law: $\vec{F} = q(\vec{E} + \vec{v} \wedge \vec{B})$. Typical problems: velocity selector, circular motion, mass spectrograph, cyclotron.

(v) Force on a current carrying wire: $d\vec{F} = id\vec{l} \wedge \vec{B} = ilBsin(\theta)$, direction from RHR. Typical problems: DC electric motor (Faraday motor). Proof that if you use $\vec{j} = nq\vec{v}$, and $i = \int \vec{j} \cdot d\vec{a}$, then this force reduces to the Lorentz force law, $\vec{F} = q\vec{v} \wedge \vec{B}$ for each charge carrier in the current carrying wire. Show that this formula leads to the force between two current carrying wires $F/L = \mu_0 i_1 i_2/(2\pi d)$.

(vi) In magnetostatics the magnetic field is divergence free, and we have the vector identity $\vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{F}) = 0$ for any vector function $\vec{F}$, therefor if we write $\vec{B} = \vec{\nabla} \wedge \vec{A}$, then we ensure that the magnetic field is divergence free. This still allows quite a bit of freedom, which is removed by choosing a gauge. In magnetostatics, the most useful gauge is the coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$. With this choice we find that, $\nabla^2 \vec{A} = -\mu_0 \vec{j}$.

B. Magnetostatics of Magnetic Materials

(i) Magnetisation and related quantities: In magnetic materials, we define magnetization, $\vec{M}$ to be the density of aligned magnetic dipoles. The relations between bound currents and the magnetization are,

$$\vec{K}_b = \vec{M} \wedge \vec{n}; \quad \vec{j}_b = \vec{\nabla} \wedge \vec{M} \tag{3}$$

With these definitions, Ampere's law becomes,

$$\vec{\nabla} \wedge \vec{B} = \mu_0(\vec{j}_f + \vec{j}_b) = \mu_0(\vec{j}_f + \vec{\nabla} \wedge \vec{M}) = \mu_0(\vec{\nabla} \wedge \vec{H} + \vec{\nabla} \wedge \vec{M}) \tag{4}$$

where $\vec{H}$ is the magnetic intensity and obeys,

$$\vec{\nabla} \wedge \vec{H} = \vec{j}_f \quad \text{so that} \quad \oint \vec{H} \cdot d\vec{l} = i_f; \quad \text{and} \quad \vec{B} = \mu_0(\vec{H} + \vec{M}) \tag{5}$$

From this equation it is seen that Amperian contours can be used to relate i_f to the magnetic intensity (or Auxilliary field) $\vec{H}$. The magnetic intensity and magnetization have units Amp per meter (A/m) as can be seen from the definition of $\vec{H}$. Typical problems: Magnetic field inside and outside a uniformly magnetized sphere or cylinder. Magnetic field a long way from a magnetized object (use dipole formula with $\vec{m} = \vec{M} \times volume$).

(ii) Linear magnetic materials, $\vec{M} = \chi_m \vec{H}, \vec{B} = \mu \vec{H} = \mu_0(1 + \chi_m)\vec{H}$. Typical problems: Ampere's law problems for sheet of current, wires/cylinders or current, solenoid containing magnetic material - if you are given a free current, find $\vec{H}$ using $\oint \vec{H} \cdot d\vec{l} = i_f$, and then find the other quantities of interest using the formulae above. Problems where there is a cylinder or sphere in a uniform magnetic field, but no free current. If there is no free current, we can introduce the scalar potential, $\vec{H} = -\vec{\nabla}\phi_m$, and in materials where χ_m is constant, we then have $\nabla^2 \phi_m = 0$. The magnetic field inside is constant and outside is like a dipole plus the applied field. The boundary conditions are $H_t^{ext} - H_t^{int} = 0$ and $B_n^{ext} - B_n^{int} = 0$.

C. Faraday's law, inductance and mutual inductance

A time varying magnetic flux leads to an emf,

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial \phi_B}{\partial t}; \quad \text{or} \quad \vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{6}$$

Typical problems: Motional emf - loop moving through a constant magnetic field. A wire moving through a constant magnetic field. Understand the correspondence between Faraday's law and the Lorentz force law for a rod moving through a constant magnetic field. A loop rotating through a constant B field leads to an AC current - an AC generator.

Definitions of self-inductance $Li = N\phi$, and mutual inductance $Mi = N\phi$, where the latter applies to a loop of N turns that is not connected to the wire that carries current i. Typical problems: A loop near a current carrying wire (mutual inductance), the inductance of a solenoid (self-inductance). Energy stored in an inductor is $Li^2/2$ and hence $u = B^2/(2\mu_0)$.

D. Maxwell's displacement current

When a capacitor is charging a surface that lies between the capacitor plates can be used, so that Amperes law of magnetostatics states that there is no enclosed current and hence the magnetic field is zero. This is wrong. There is a magnetic field produced by the current. Maxwell resolved this difficulty by adding a new term which includes the effect of the electric field which builds up between the capacitor plates. His idea was to related this electric field to the current flowing the circuit. From Gauss' law, we have,

$$\frac{d\phi_E}{dt} = \frac{1}{\epsilon_0}\frac{dq}{dt} \tag{7}$$

or the "Maxwell displacement current" is,

$$i_d = \epsilon_0 \frac{d\phi_E}{dt} \tag{8}$$

This relates the current flowing into the capacitor to the electric field between the plates. Maxwell realized that if Amperes law is modified to,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0(i + i_d) = \mu_0 i + \mu_0\epsilon_0 \frac{d\phi_E}{dt} \tag{9}$$

then the logical inconsistency in the case of a charging capacitor is removed.

E. The wave equations and EM waves

(i) Wave solutions: From Maxwell's equations in vacuum we find that,

$$\nabla^2 \vec{E} = \mu_0\epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \tag{10}$$

$$\nabla^2 \vec{B} = \mu_0\epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \tag{11}$$

These are both wave equations. A simple linearly polarized wave solution is,

$$\vec{E}(x,y,z) = E_0 cos(kz - \omega t + \phi)\hat{\imath}; \quad \vec{B}(x,y,z) = B_0 cos(kz - \omega t + \phi)\hat{\jmath}. \tag{12}$$

The magnitudes of the electric and magnetic fields are related by, $|\vec{E}| = c|\vec{B}|$ and the velocity of light is related to the permeability and permitivity through, $c = (\mu_0\epsilon_0)^{-1/2}$. Typical problems: Demonstate that the wave equations follow from Maxwell's equations in free space.

(ii) The energy flux density is given by Poynting's vector $\vec{S} = \frac{1}{\mu_0}\vec{E} \wedge \vec{B}$. The energy density is given by $u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2 = \epsilon_0 E^2 = \frac{B^2}{\mu_0} = \frac{EB}{c\mu_0}$. These two quantities are related by $|S| = uc$. The intensity is the time averaged energy flux density, so $I = S_{ave}$. In quantum mechanics each photon has energy $h\nu$, so the intensity is $I = n(t)h\nu$ where $n(t)$ is the number of photons arriving per unit time per unit area.