PHY481 - Lecture 19: The vector potential, boundary conditions on $\vec{A}$ and $\vec{B}$.
Griffiths: Chapter 5

The vector potential

In magnetostatics the magnetic field is divergence free, and we have the vector identity $\vec{\nabla}\cdot(\vec{\nabla}\wedge\vec{F}) = 0$ for any vector function $\vec{F}$, therefore if we write $\vec{B} = \vec{\nabla}\wedge\vec{A}$, then we ensure that the magnetic field is divergence free. $\vec{A}$ is the vector potential, and despite being a vector simplifies the calculations in some cases. It is also very important in quantum mechanics where the solution to Schrodinger's equation in a magnetic field involves adding a term proportional to $\vec{A}$ to the momentum operator. To find a differential equation for the vector potential, we use the differential form of Ampere's law to find,

$$\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{A}) = \mu_0\vec{j} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} \tag{1}$$

where we used a vector identity to write the last expression on the RHS. This still looks messy, however now note that the choice of $\vec{A}$ is not unique as we can write $\vec{A} + \vec{\nabla}f$ and $\vec{\nabla}\wedge\vec{A}$ is unaltered. We can then choose the scalar function f to help solve problems. In electrostatics a convenient choice is $\vec{\nabla}\cdot\vec{A} = 0$ as this removes the first term in the last expression on the RHS of Eq. (1). This choice is called the Coulomb gauge, and in this gauge we have,

$$\nabla^2\vec{A} = -\mu_0\vec{j}; \quad \text{Coulomb gauge.} \tag{2}$$

This is just Poisson's equation for each of the components of $\vec{A}$. Recall that for the voltage $\vec{\nabla}^2V = -\rho/\epsilon_0$. To show that it is always possible to find a scalar function f to ensure that $\vec{\nabla}\cdot\vec{A} = 0$, consider a problem where we have found a solution $\vec{A}_s$ where $\vec{\nabla}\cdot\vec{A}_s \neq 0$. We then need to find a scalar function f_s so that $\vec{\nabla}\cdot(\vec{A}_s + \vec{\nabla}f_s) = 0$. This reduces to $\nabla^2 f_s = -\vec{\nabla}\cdot\vec{A}_s$. This is just Poisson's equation again. It always has a solution, so we can always find a scalar f_s so that $\vec{A} = \vec{A}_s + \vec{\nabla}f_s$ is divergence free.

Now that we have found that the vector potential in the Coulomb gauge obeys Poisson's equation, the solution to these equations in integral form may be written down,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{j}(\vec{r}')d^3r'}{|\vec{r}-\vec{r}'|}; \quad \text{while in electrostatics} \qquad V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}')d^3r'}{|\vec{r}-\vec{r}'|}. \tag{3}$$

A very important integral relation between $\vec{A}$ and $\vec{B}$

The vector potential is related to the magnetic flux through,

$$\oint\vec{A}\cdot d\vec{l} = \int(\vec{\nabla}\wedge\vec{A})\cdot d\vec{a} = \int\vec{B}\cdot d\vec{a} = \phi_B. \tag{4}$$

We also know that the direction of A is the same as the direction of the current density, so we can find $\vec{A}$ from the magnetic field in symmetric cases.

Aharanov-Bohm effect

The integral of the potential arises in many contexts in quantum mechanics and leads to interference effects in mesoscopic conductors and in superconductors. For example the critical current of a squid is used to measure magnetic flux using this effect.

Boundary conditions on the magnetic field and vector potential

Recall the boundary conditions across a charged surface in electrostatics,

$$V_{above} - V_{below} = 0; \quad E^{\parallel}_{above} - E^{\parallel}_{below} = 0; \quad E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0} \tag{5}$$

which follow from the integral forms,

$$V_{ab} = -\int\vec{E}\cdot d\vec{l}; \quad \oint\vec{E}\cdot d\vec{l} = 0; \quad \oint\vec{E}\cdot d\vec{a} = q/\epsilon_0 \tag{6}$$

The boundary conditions across a current carrying surface in magnetostatics are,

$$\vec{A}_{above} - \vec{A}_{below} = 0; \quad B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K; \quad B^{\perp}_{above} - B^{\perp}_{below} = 0 \tag{7}$$

These equations follow from the integral forms,

$$\oint \vec{A} \cdot d\vec{l} = \phi_B; \quad \oint \vec{B} \cdot d\vec{l} = \mu_0 i; \quad \oint \vec{B} \cdot d\vec{a} = 0 \tag{8}$$

The derivation is analogous to the electrostatic case.

Calculating the vector potential

To calculate the vector potential, we can use Eq. (2) or Eq. (3), or we can use the integral form (4). The latter is the easiest and works in symmetric cases where Ampere's law can be used to find the magnetic field, as follows.

Solenoid:

Consider the simplest of the magnetic field inside a solenoid, where $\vec{B} = \mu_0 ni$, when the solenoid axis is along the $\hat{z}$ direction. The current is in the $\hat{\phi}$ direction. To use Eq. (), we have to choose a contour and clearly the symmetric choice here is a circle with normal in the $\hat{z}$ direction. The flux through the circle depends on the radius of the circle, s. For a solenoid of radius R, the flux for $s < R$ is $\phi_B(s < R) = Ba(s) = \mu_0 ni\pi s^2$; while for $s > R$, we have $\phi_B(s > R) = \mu_0 ni\pi R^2$. Assuming that the magnitude of $\vec{A}$ only depends on s, we then find,

$$\oint \vec{A} \cdot d\vec{l} = 2\pi s A(s) = \phi_B(s) \quad \text{so} \quad A(s < R) = \frac{\mu_0 nis}{2}; \quad A(s > R) = \frac{\mu_0 niR^2}{2s} \tag{9}$$

Infinite wire: In the case of an infinite wire that carries current $i\hat{z}$, we assume that $\vec{A}$ is also in the $\hat{z}$ direction. We therefore choose a rectangular contour with one side parallel to the wire and at distance s and the other at position a. The flux in the loop is then,

$$\phi_B = \int_s^a \frac{\mu_0 i}{2\pi s} ds = \frac{\mu_0 i}{2\pi} ln(a/s) \tag{10}$$

The contour integral gives $A(s)L - A(a)L$. We then deduce that $A(s) = -\frac{\mu_0 i}{2\pi} ln(s)$, so the vector potential is $\vec{A} = -\frac{\mu_0 i}{2\pi} ln(s)\hat{z}$. Taking the curl confirms that this reproduces the correct magnetic field. This expression has all of the gauge degrees of freedom removed.

Current sheet

The third case that we looked at was an infinite sheet with current density $\vec{K} = K\hat{x}$ lying in the x-y plane. What is the vector potential in that case? Using Ampere's law, we find that the magnetic field for $z > 0$ is $\vec{B}(z > 0) = -\mu_0 K\hat{y}/2$. Taking a rectangular contour with normal $-haty$, we find,

$$[A(z) - A(a)]L = \frac{\mu_0 KL}{2}(a - z) \tag{11}$$

so that $\vec{A}(z) = -\frac{\mu_0 Kz}{2}\hat{x}$. The gauge degrees of freedom have been removed from this expression.

Freedom in the vector potential

To illustrate the freedom in choosing the vector potential, consider the case of a constant magnetic field in $\hat{z}$ direction $\vec{B} = B_0\hat{z}$. A vector potential that corresponds to this magnetic field is $\vec{A} = \frac{1}{2}(-B_0 y, B_0 x, 0)$, but $\vec{A} = (-B_0 y, 0, 0)$ also works. In fact if we write down the three equations corresponding to $B_0\hat{z} = \vec{\nabla} \wedge \vec{A}$ in Cartesian co-ordinates, we have,

$$\frac{\partial A_z}{\partial y} = \frac{\partial A_y}{\partial z}; \quad \frac{\partial A_x}{\partial z} = \frac{\partial A_z}{\partial x}; \quad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_0; \tag{12}$$

A solution of the form $\vec{A} = (-ay, bx, f(z))$ with $a + b = B_0$ satisfies these equations. If we impose the Coulomb Gauge condition we have $\partial f/\partial z = 0$. Clearly there are many degrees of freedom in our choice of the vector potential, all of which should lead to the same physics, e.g. the same magnetic field.