

## PHY481 - Lecture 3: Vector calculus

Griffiths: Chapter 1 (Pages 38-54), Also Appendix A of Griffiths

### Vector calculus in three co-ordinate systems

We shall be using three orthogonal co-ordinate systems, cartesian, cylindrical and spherical polar that are defined as follows

1. Cartesian co-ordinates

$$\vec{r} = (x, y, z) = x\hat{x} + y\hat{y} + z\hat{z} \quad (1)$$

2. Cylindrical co-ordinates

$$\vec{r} = (s, \phi, z) = s\hat{s} + z\hat{z} = s\cos(\phi)\hat{x} + s\sin(\phi)\hat{y} + z\hat{z} \quad (2)$$

where,

$$x = s\cos(\phi); \quad y = s\sin(\phi); \quad x^2 + y^2 = s^2 \quad (3)$$

3. Spherical polar co-ordinates

$$\vec{r} = (r, \theta, \phi) = r\hat{r} = r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y} + r\cos(\theta)\hat{z} \quad (4)$$

where,

$$x = r\sin(\theta)\cos(\phi); \quad y = r\sin(\theta)\sin(\phi); \quad z = r\cos(\theta); \quad x^2 + y^2 + z^2 = r^2 \quad (5)$$

### Unit vector transformations

Relations between unit vectors in the three co-ordinate systems are very useful and can be derived using geometric reasoning. First consider cylindrical co-ordinates. It is easy to see that,

$$\hat{s} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y}; \quad \hat{\phi} = -\sin(\phi)\hat{x} + \cos(\phi)\hat{y} \quad (6)$$

The first of these equations follows from Eq. (2) and the second is evident on taking projections of  $\hat{\phi}$  onto the x and y directions. It is also easy to find the transformations from cylindrical to cartesian unit vectors;

$$\hat{x} = \cos(\phi)\hat{r} - \sin(\phi)\hat{\phi}; \quad \hat{y} = \sin(\phi)\hat{r} + \cos(\phi)\hat{\phi} \quad (7)$$

The transformations in the case of polar co-ordinates are,

$$\hat{r} = \sin(\theta)\cos(\phi)\hat{x} + \sin(\theta)\sin(\phi)\hat{y} + \cos(\theta)\hat{z}; \quad (8)$$

which follows from Eq. (4),

$$\hat{\theta} = \cos(\theta)\cos(\phi)\hat{x} + \cos(\theta)\sin(\phi)\hat{y} - \sin(\theta)\hat{z}; \quad (9)$$

and

$$\hat{\phi} = -\sin(\phi)\hat{x} + \cos(\phi)\hat{y} \quad (10)$$

which is the same as the second of Eq. (6). The equations relating unit vectors in spherical polars to those in cartesian co-ordinates are,

$$\hat{x} = \sin(\theta)\cos(\phi)\hat{r} + \cos(\theta)\cos(\phi)\hat{\theta} - \sin(\phi)\hat{\phi}; \quad (11)$$

$$\hat{y} = \sin(\theta)\sin(\phi)\hat{r} + \cos(\theta)\sin(\phi)\hat{\theta} + \cos(\phi)\hat{\phi}; \quad (12)$$

and

$$\hat{z} = \cos(\theta)\hat{r} - \sin(\theta)\hat{\theta} \quad (13)$$

**Curvilinear co-ordinates: Scale factors**  $h_1, h_2, h_3$

In general a set of curvilinear co-ordinates can be orthogonal or non-orthogonal. We focus on the orthogonal case, which includes cartesian, cylindrical and spherical co-ordinates. We denote the unit vectors as  $\hat{e}_1, \hat{e}_2, \hat{e}_3$ , and a position vector  $\vec{l}$  is written as,

$$\vec{l} = (u_1, u_2, u_3) = u_1\hat{e}_1 + u_2\hat{e}_2 + u_3\hat{e}_3. \quad (14)$$

For the Cartesian, Cylindrical, and Spherical Polar cases, we may write  $(u_1, u_2, u_3) = (x, y, z), (s, 0, z), (r, 0, 0)$  respectively.

Now imagine displacing the co-ordinates by a small amount  $du_1, du_2, du_3$ , this leads to a change in the vector  $\vec{l}$  by an amount  $d\vec{l}$ . In general we can write

$$d\vec{l} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3 \quad (15)$$

where  $h_1, h_2, h_3$  are the scale factors. They are central to deriving the relations that we need. For the three cases of interest in this course, we have,

$$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}; \quad d\vec{l} = ds\hat{s} + sd\phi\hat{\phi} + dz\hat{z} \quad d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin(\theta)d\phi\hat{\phi} \quad (16)$$

leading to: For *cartesian co-ordinates*,  $h_x = 1, h_y = 1, h_z = 1$ ; for *cylindrical co-ordinates*,  $h_s = 1, h_\phi = s, h_z = 1$ ; and for *spherical polar co-ordinates*,  $h_r = 1, h_\theta = r, h_\phi = r\sin\theta$ .