Quantum theory of transitions between stable states of a nonlinear oscillator interacting with a medium in a resonant field

M. I. Dykman and V. N. Smelyanskiĭ

Institute of Semiconductors of the Ukranian Academy of Sciences

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A general expression for the probability of transitions between stable states of an oscillator in a strong resonant field, with a frequency significantly exceeding the reciprocal relaxation time, is obtained in a quasiclassical approximation. The transition probability is determined to logarithmic accuracy by the action for a certain auxilary system moving in a complex phase space, while the time component is real. It is shown that in the quasi-energy representation the transitions are of the "above-the-barrier" (not tunnel) kind, regardless of the temperature of the medium. They are attributable to diffusion over the quasi-energy, due to the relaxation processes. Explicit expressions for the transition probabilities are obtained for the case of weak damping. Their dependence on the parameters of the oscillator and on the temperature of the medium is investigated.

INTRODUCTION

Many physical systems have several stable states. If the quantum diffusion and fluctuations, arising from interactions between a system and the medium (thermostat), are small enough, then the system is predominantly localized in the close vicinity of the stable states. The transitions between the states occur infrequently: their probabilities W per unit time are small compared with the characteristic reciprocal relaxation time τ_r^{-1}. When these systems are in thermodynamic equilibrium, the transitions are caused by tunneling and by thermal activation; their statistical distribution is always of the Gibbs type, regardless of the mechanism of interactions with the medium.

The nature of the transitions and the form of the distribution turn to be more complex for systems, placed in a strong field that is periodic in time (and in general, for the systems not at equilibrium). Even when $T = 0$ and the equilibrium system is localized at the lowest energy level, and even for weak interaction with the medium, the quasi-energy distribution has, generally speaking, a finite width (when describing the quantum states in a periodic field, the quasi-energy plays a role analogous to that of the energy of the systems at equilibrium[1]). Indeed, even though only relaxation processes with the transfer of energy from the system to thermostat take place when $T = 0$, in reality an elementary scattering act may cause the system's quasi-energy to approach its value for the stable state, but also to move away from that value, with some finite relative probability $\omega < 1$ (because quantum states with definite quasi-energy are superpositions of states with definite energy).

Thus, even when $T = 0$ there is a probability of "above-the-barrier" (in the quasi-energy representation) transition from a stable state, induced by dissipation. In a quasiclassical case, when the number N of quasi-energy states localized in the proximity of a given stable state is large, the probability for this transition is

$$W \propto \exp(-A/\hbar), \quad A \sim \hbar N |\ln w|.$$

We note that A does not include the constant of interaction with the medium in the limit of weak interaction.

If, in addition to the quantum states localized in the vicinity of the stable state, there exist quantum states localized elsewhere but with the same values of quasi-energy (metastability in the quasi-energy representation), the direct tunnelings from the stable states are also possible. Thermal fluctuations also affect the probabilities of transitions at finite temperatures. The prevalence of either transition mechanism is determined not only by the properties of an isolated system and the temperature (as in equilibrium systems with weak dissipation), but also by the mechanism of interaction with the medium. The latter also determines the form of the system's distribution in quasi-energy.

The problem of transitions between stable states, when quantum effects are neglected, was considered in a number of works, beginning with Ref. 2. In this approximation, the transition probability W calculated with logarithmic accuracy is equal to the probability of the optimal fluctuation that takes the system from one state to another along a certain trajectory in the phase space. The calculation of $\ln W$ in the general case reduces to the computation of this trajectory (cf. Refs. 3 and 4). If quantum effects are essential, then, generally speaking, the notion of the optimal trajectory of transition in real phase space is inadequate. Specifically, the momentum is imaginary for "pure" tunneling.[5]

It is shown below that in some cases a transition in a system interacting with a medium, can be related to the motion of an auxiliary dynamic system with twice as many degrees of freedom as the original system. This motion takes place in a complex phase space and in real time. Its characteristic duration is determined by a natural parameter, the relaxation time τ_r.

The concept of motion with real time is especially important when describing systems placed in a strong high frequency field, when the height of the quasi-energy barrier is $\Delta U \ll \hbar\omega$. In such systems $\tau_t \gg \omega^{-1}$, where $i\tau_t$ is the imaginary tunneling time defined in the standard manner. It is easy to see that within a "time" $i\tau_t$ even weak interaction with the medium $\tau_r^{-1} \ll \Delta U/\hbar$ (but $\tau_r^{-1} > W_t$, where W_t is the probability of tunneling) would have caused a complete change in the character of the motion: the correction to the quasi-energy due to the oscillating (in real time) terms in the Hamiltonian of interaction would have become $\sim \hbar\tau_r^{-1} \exp(\omega\tau_t)$, which greatly exceeds the barrier height

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$\Delta U(|\ln W_t| \sim \Delta U \tau_t/\hbar$, and for that reason, $\tau_r^{-1} > W_t \gg (\Delta U/\hbar) \exp(-\omega \tau_t)$.

In the present paper the problem of computations of the transitions probability and of the quasistationary distribution is considered for a nonlinear oscillator excited by a resonant field. In the sufficiently strong field, the nonlinear oscillator can have[6] two stable states with different amplitudes of the forced oscillations. This system is of immediate interest in connection with the well-known problem of collisionless dissociation of molecules in a laser field.[7] It is used as a model when investigating optical bistability (see Refs. 8–10). The nonlinear-oscillator model describes also the lateral motion of an electron with nonparabolic dispersion in a magnetic field. According to recent experimental data[11] bistability of the cyclotron motion was observed under resonant pumping.

The theory of the fluctuation transitions between stable states of a classical oscillator in a resonant field was developed in Refs. 3 and 12. The limiting case of small dissipation was considered also by another method in Refs. 13 and 14. Calculations of tunneling probabilities for an oscillator were carried out in Refs. 15 and 16, and most completely in Ref. 13. The stationary distribution for certain values of parameters was calculated numerically in Ref. 17.

The kinetics of the oscillator is analyzed below, in the quasiclassical approximation. The transition probabilities are assumed exponentially small and calculated to logarithmic accuracy. In chapter 1 the quantum kinetic equation is derived. In chapter 2 it is solved utilizing the method of eikonals, and a general expression for the probability of transitions per unit time is given. In chapter 3 the complex extremal trajectory of the auxilliary system is found for the case of weak damping. In chapter 4 it is used for the calculations of the transition probabilities and for the analysis of some extremal cases. In chapter 5 the probability of leaving a state with smaller oscillations amplitude is investigated in weak fields. The Conclusion contains some summarizing remarks.

1. KINETIC EQUATION FOR A NONLINEAR OSCILLATOR

The hamiltonian of the isolated oscillator is

$$H_0 = \tfrac{1}{2}(p_0^2 + \omega_0^2 q_0^2) + \tfrac{1}{4}\gamma q_0^4 - q_0 F \cos \omega_F \tau. \tag{1}$$

Expression (1) is written for the simplest model of an oscillator (Duffing model), where the dependence of a natural frequency on the amplitude arises as early as in the first order in anharmonicity. It is precisely this expression which leads for $\gamma(\omega_F - \omega_0) > 0$ to bistability in a resonant field $F \cos \omega_F \tau$ in a definite interval of F.[6]

If the anharmonicity and the interaction with the medium are small enough, then the motion of an oscillator can be divided into "fast" (with frequencies which are multiples of ω_F) and "slow." The latter can be conveniently described in terms of the smooth, dimensionless, dynamic variables q and p:

$$q = (\lambda\omega_F/\hbar)^{1/2}(q_0 \cos \omega_F \tau - p_0 \omega_F^{-1} \sin \omega_F \tau),$$
$$p = (\lambda\omega_F/\hbar)^{1/2}(q_0 \sin \omega_F \tau + p_0 \omega_F^{-1} \cos \omega_F \tau), \quad [p, q] = -i\lambda,$$
$$\lambda = 3\hbar|\gamma|(8\omega_F^2|\delta\omega|)^{-1}, \quad \delta\omega = \omega_F - \omega_0, \quad |\delta\omega| \ll \omega_F. \tag{2}$$

The parameter λ is determined by the ratio of the difference in the adjacent frequencies of transitions between the oscillator's energy levels to the detuning of the field frequency $\omega_F - \omega_0$. We assume

$$\lambda \ll 1. \tag{3}$$

When Eq. (3) is satisfied, the slow motion is almost invariably quasiclassical (cf. Ref. 13); λ plays the role of the Planck's constant for the slow motion.

Neglecting small rapidly oscillating terms we obtain from Eqs. (1) and (2) the equation of motion for q and p

$$\dot q = dq/dt = \partial g/\partial p, \quad \dot p = dp/dt = -\partial g/\partial q,$$
$$t = \tau\delta\omega \quad (\delta\omega > 0, \quad \gamma > 0), \tag{4}$$

where

$$g \equiv g(q, p) = \tfrac{1}{4}(q^2 + p^2 - 1)^2 - q\beta^{1/2}, \quad \beta = 3|\gamma| F^2 (32\omega_F^3 |\delta\omega|^3)^{-1} \tag{5}$$

(the products of the operators p and q have to be symmetrized; to be specific, we assume $\delta\omega > 0$ and $\gamma > 0$). The dimensionless operator $g(q, p)$ in Eq. (4) determines the spectrum of the quasi-energies of the system. It contains only one dimensionless parameter β which describes the intensity of the resonant field.

If the interaction between the system and the oscillator is so weak that there is very small damping of oscillations within one period $2\pi/\omega_0$, then the envelope of the oscillator's density matrix ρ for the times $\Delta\tau \gg \omega_0^{-1}, \omega_m^{-1}$ $\hbar\omega_m$ is the characteristic excitation energy for the medium) can be described by an equation without delay (cf. Ref. 18)

$$\dot\rho(q_1, q_2) = -i\lambda^{-1}\mathscr{H}(q_1, q_2, -i\lambda\partial/\partial q_1, -i\lambda\partial/\partial q_2)\rho(q_1, q_2), \tag{6}$$
$$\mathscr{H}(q_1, q_2, p_1, p_2) = g(q_1, p_1) - g(q_2, p_2) + \mathscr{R}(q_1, q_2, p_1, p_2).$$

In many oscillating systems the relaxation is due to an interaction, linear to the system's coordinate (momentum), with the medium. Consequently,

$$\mathscr{R}(q_1, q_2, p_1, p_2) = \Omega^{-1} R(q_1, q_2, p_1, p_2), \quad \Omega = \delta\omega/\Gamma,$$
$$R = -i(\bar n + \tfrac{1}{2})[(q_1 - q_2)^2 + (p_1 + p_2)^2] - (p_1 q_2 + p_2 q_1), \tag{7}$$
$$\bar n = [\exp(\hbar\omega_F/T) - 1]^{-1}.$$

Expression (7), with Eq. (2) taken into consideration, corresponds to the expression, well known for an oscillator, for the linear friction operator in the occupation-number representation (cf. Ref. 18; the term $i\lambda$ is omitted from R, since it is small wherever the quasiclassical approximation is applicable). The damping parameter Γ corresponds to the friction coefficient in the phenomenological description of relaxation (the friction force equals $-2\Gamma dq_0/d\tau$). When deriving (6) and (7) it was assumed, that Γ, $\delta\omega$, $\lambda\delta\omega \cdot (2\bar n + 1) \ll \omega_0$, ω_m (see Ref. 18 for details). At the same time, the ratio of Γ and $\delta\omega$ (the parameter Ω^{-1}) can be arbitrary, i.e., in a certain sense, the damping is not considered weak.

Equation (6) can also be rewritten in the Wigner representation

$$\dot\rho_W(q, p) = -i\lambda^{-1}\mathscr{H}(q + \tfrac{1}{2}i\lambda\partial/\partial p, \ q - \tfrac{1}{2}i\lambda\partial/\partial p,$$
$$p - \tfrac{1}{2}i\lambda\partial/\partial q, \ -p - \tfrac{1}{2}i\lambda\partial/\partial q)\rho_W(q, p),$$
$$\rho_W(q, p) = \int d\xi \exp(-i\lambda^{-1}\xi p)\rho(q + \tfrac{1}{2}\xi, q - \tfrac{1}{2}\xi). \tag{8}$$

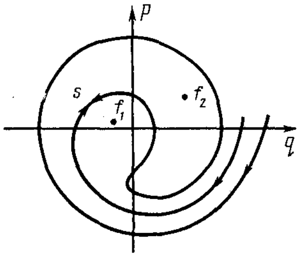

FIG. 1. Phase plane for the classical nonlinear oscillator (q and p are the slow variables [cf. Eq. (2)]. The solid line is the separatrix between the regions of attraction to the foci f_1 and f_2, while S is the saddle point.

The solution of (8) in the lowest order in λ describes the motion of an oscillator along the classical trajectory

$$\dot{q}=\partial g/\partial p-\Omega^{-1}q, \ \dot{p}=-\partial g/\partial q-\Omega^{-1}p. \tag{9}$$

Equation (9) has three stationary solutions for the following intervals of parameters $\Omega > 3^{1/2}$, $\beta_B^{(1)}(\Omega) < \beta < \beta_B^{(2)}(\Omega)$ (the values of $\beta_B^{(1,2)}$ are given in Ref. 3). Two of the solutions (with the minimum and maximum values of the sum q^2+p^2) correspond to the stable state, and the third solution corresponds to the unstable state of the oscillator. The foci (nodes) f_1, f_2 and the saddle point S on the phase plane correspond to these states. The separatrix passes through the point S, which separates the areas of "attraction" to the foci. The form of the phase plane is depicted in Fig. 1 for the case of relatively weak damping.

2. GENERAL EXPRESSION FOR THE TRANSITION PROBABILITY

When $\lambda \ll 1$ the system, initially localized in the vicinity of the point of common position in the phase space, has during the dimensionless relaxation time Ω^{-1}, an overwhelming probability, after following the classical trajectory (9) of eventually approaching that stable state (state 1, for example) in whose domain of attraction it was initially positioned. The transitions between states occur after much longer times ($\sim W^{-1} \propto \exp(G/\lambda)$). It is obvious that the probability for the transition $1 \to 2$ in unit time, W_{12}, is the same as the probability to reach the domain of attraction of state 2 in unit time. Having reached that region at some point, the system subsequently approaches the state 2, following the trajectory (9).

The probabilities to reach different points which are not located on the same trajectory (9) and are separated from each other by $\Delta q, \Delta p \gg [\lambda(2\bar{n}+1)]^{1/2}$, are significantly different. To logarithmic accuracy, the value of W_{12} is determined by the probability to reach a certain optimal point, the point of "entry." In the classical limit ($\lambda \to 0$, $\bar{n} \to \infty$, $\lambda\bar{n} \ll 1$), when the bundle of transition trajectories in the phase space is narrow, the point of "entry" is obviously located on the boundary (the saddle point is the "entry," cf. Ref. 3). For tunneling, it is located inside the region of attraction of the state 2.

In the time interval $\Omega^{-1} \gg t \gg W_{12}^{-1}$ the distribution of the system $\rho_W(q,p)$ is quasistationary almost everywhere in phase space, except for the narrow neighborhood ($\Delta q, \Delta p \sim [\lambda(2\bar{n}+1)]^{1/2}$ of state 2, where the population rises linearly with time, due to the flow from state 1. Virtually the entire value of $\rho_W(q,p)$ is located in the vicinity of state 1. It is exponentially small in the region of attraction of state 2, but it has a sharp maximum in that part of the trajectory (9) which leads from the point of "entry" to state 2. To logarithmic accuracy, $\rho_W(q,p)$ is constant along this portion of the trajectory and it is equal to W_{12}.

It is convenient to look for a quasistationary solution of the equation (6) in a quasiclassical form[1)]

$$\rho(q_1, q_2)=\rho_0 \exp [i\lambda^{-1}S(q_1, q_2)], \tag{10}$$

where the function S satisfies the equation

$$\mathscr{H}\left(q_1, q_2, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}\right)=0, \tag{11}$$

and ρ_0 determines a relatively smooth pre-exponential factor that will not be investigated in the present work. If the system is localized predominantly in the vicinity of a focus (node) f with the coordinates q_f and p_f on the phase plane, i.e., $\rho_W(q_f, p_f) \approx 1$, then it follows from Eqs. (6)–(8) and (10) that

$$S(q_f, q_f)=0, \ (\partial S/\partial q_1)_{q_1=q_2=q_f}=-(\partial S/\partial q_2)_{q_1=q_2=q_f}=p_f. \tag{12}$$

The foregoing quantitative considerations lead to the conclusion that in the region $\Omega^{-1} \ll t \ll W^{-1}$ the probability of a transition $f \to f'$ in unit time is

$$W=\text{const} \exp(-\lambda^{-1}G), \quad G=\min \operatorname{Im} S(\tilde{q}, \tilde{q}),$$
$$(\partial S/\partial q_1)_{q_1=q_2=\tilde{q}}=-(\partial S/\partial q_2)_{q_1=q_2=\tilde{q}}=\tilde{p}, \tag{13}$$

where the minimum is taken over the points $\tilde{q}$ and $\tilde{p}$ in the attraction region of the state f', including its boundary. Thus, the calculation of $\ln W$ are reduced to computation of the function S satisfying Eq. (11), with the boundary conditions (12) and (13). The criterion of applicability of Eq. (13) is

$$G \gg \lambda. \tag{14}$$

Equation (12) can be viewed as the Hamilton-Jacobi equation for some auxiliary classical two-dimensional particle with the coordinates q_1 and q_2, with $S(q_1,q_2)$ its action, and $\mathscr{H}(q_1,q_2,p_1,p_2)$ the Hamilton function (cf. Ref. 6). The equations of motion for this particle are

$$\dot{q}_j=\partial\mathscr{H}/\partial p_j, \quad \dot{p}_j=-\partial\mathscr{H}/\partial q_j \quad (j=1,2),$$
$$S(t)=\int dt\,(p_1\dot{q}_1+p_2\dot{q}_2), \tag{15}$$

and the particle moves with the energy $\mathscr{H}=0$. The trajectory which determines the probability of transitions from the focus f corresponds to the boundary conditions

$$q_1(0)=q_2(0)\approx q_f, \quad q_1(t)=q_2(t)\approx\tilde{q}. \tag{16}$$

The momentum components corresponding to the initial and final points are determined by the values of $q_{1,2}(0)$ and $q_{1,2}(t)$; the values of $p_j(0)$ are close to $(-1)^{j+1}p_f$ while $p_j(t)$ are close to $(-1)^{j+1}\tilde{p}$ ($j=1,2$). It is not difficult to find the values of $p_{1,2}(0)$ if Eqs. (15) are linearized in the vicinity of the focus. Then, taking Eq. (12) into account, we have

$$p_j(0)=(-1)^{j+1}p_f+\sum_{j'}A_{jj'}[q_{j'}(0)-q_f] \quad (j,j'=1,2), \tag{17}$$

where $A_{jj'}$ are determined by the equations

$$\frac{1}{2}\frac{\partial^2\mathscr{H}}{\partial q_j\,\partial q_{j'}}+\sum_{j_1}\frac{\partial^2\mathscr{H}}{\partial q_j\,\partial p_{j_1}}A_{j_1j'}+\frac{1}{2}\sum_{j_1j_2}\frac{\partial^2\mathscr{H}}{\partial p_{j_1}\,\partial p_{j_2}}A_{j_1j}A_{j_2j'}=0 \tag{18}$$

(the derivatives in (18) are calculated in the focus). Equation (17) corresponds to allowance for the quadratic terms in the equation for the action $S(q_1,q_2)$ in the vicinity of the focus,

$$S\approx p_f(q_1-q_2)+\frac{1}{2}\sum_{jj'}A_{jj'}(q_j-q_f)(q_{j'}-q_f),\qquad |q_{1,2}-q_f|\ll 1, \tag{19}$$

which, in turn, corresponds to a Gaussian form of the quasistationary distribution $\rho_W(q,p)$ in the vicinity of a focus.

The solution of the kinetic equation by the eikonal method was used in many studies of fluctuations in classical systems, starting probably with Ref. 19. This method yields the same results as the method of functional integration[3] for the probabilities of transitions in the Markov system (cf. Ref. 4). It can be shown that the latter is also true for the quantum systems considered here.

We note that there are trajectories on which

$$q_2(t)=q_1^*(t),\quad p_2(t)=-p_1^*(t) \tag{20}$$

among the solutions of the Eqs. (15)–(17) (this follows from the expression $\mathscr{H}^*(q_2^*,q_1^*,-p_2^*,-p_1^*) = -\mathscr{H}(q_1,q_2,p_1,p_2)$ that follows in turn from the fact that the density matrix $\rho(q_1,q_2)$ is Hermitian). It is these trajectories, and some others close to them, which determine the values of the transition probabilities, as is clear from Eqs. (12) and (13).

Since the Hamiltonian $\mathscr{H}$ does not have singularities in the finite region $(q_{1,2},p_{1,2})$, it follows that only one extremal trajectory passes through each point $q_1=q_2=\tilde{q}$, $p_1=-p_2=\tilde{p}$ ($\operatorname{Im}\tilde{q}=\operatorname{Im}\tilde{p}=0$), except for the foci and the saddle, and in general this trajectory proceeds to the focus. The values of $q_1(t)$, $p_1(t)$ and $q_2(t)$, $-p_2(t)$ on it are described by Eq. (9) for $q(t)$, $p(t)$. With Eqs. (13), (16), and (17) taken into account, it follows that the extremal trajectory (20), along which the transition between stable states takes place, arrives at the saddle point (q_s,p_s) accurate to terms $\sim\lambda$:

$$q_1(t)=q_2(t)\approx q_s,\quad p_1(t)=-p_2(t)\approx p_s. \tag{16a}$$

This means, that in the system considered here under quasistationary conditions the transitions between states are not related to tunneling.

3. EXTREMAL TRAJECTORIES IN THE CASE OF WEAK DAMPING

The form of the trajectories and the value of G in (13) that determines the transition probability, depend on the dimensionless oscillator parameters β and Ω and on the Planck number $\bar{n}$, i.e., on the temperature. The value of G can be determined for every set of these parameters by solving numerically Eqs. (13), (15), and (17). In some limiting cases G can be found analytically. We investigate below the case of weak damping

$$\Omega^{-1}\equiv\Gamma/\delta\omega\ll 1, \tag{21}$$

when the oscillator damping is small during the characteristic period of slow oscillations $|\omega_F-\omega_0|^{-1}$. Along with the parameter $\Gamma/\delta\omega$, the type of damping is determined by the ratio of the width of the quasi-energy levels (which is $\sim\hbar\Gamma/\lambda$ in the dimensional units) to the distance between them ($\sim\hbar\delta\omega$). The corrections to the probability on the account of dissipation are small only when

$$(\lambda\Omega)^{-1}\equiv\Gamma/\lambda\delta\omega\ll 1 \tag{22}$$

(The parameter $\Gamma/\lambda\delta\omega\equiv 4\omega_0^2\Gamma/3\hbar\gamma$ also determines the ratio of the width of energy levels of a nonlinear oscillator to their nonequidistance, and determines the character of the damping in the absence of the external field, see Ref. 18).

In the limiting case of zero damping, the Hamiltonian of a two-dimensional particle $\mathscr{H}$ [Eq. (6)] is reduced to the difference $g(p_1,q_1)-g(p_2,q_2)$, i.e., the variables with the indices 1 and 2 are separable. At the same time, Eq. (15) coincides with (4) for q_1 and p_1 and differs from (4) by the sign on the right for q_2 and p_2. The solution $q(t)$ and $p(t)$ of Eqs. (4) corresponds to essentially nonlinear oscillations with fixed g. It is described by the periodic functions $Q(g,\varphi(t))$, $P(g,\varphi(t))$:

$$\begin{gathered}\frac{\partial Q}{\partial\varphi}=\omega^{-1}(g)\frac{\partial g}{\partial P},\quad \frac{\partial P}{\partial\varphi}=-\omega^{-1}(g)\frac{\partial g}{\partial Q},\\ g\equiv g(Q,P),\quad \dot{\varphi}=\omega(g),\\ Q\equiv Q(g,\varphi)=\sum_n Q_n(g)e^{in\varphi},\\ P\equiv P(g,\varphi)=\sum_n P_n(g)e^{in\varphi},\quad Q_n=Q_{-n},\quad P_n=-P_{-n}.\end{gathered} \tag{23}$$

The natural oscillation frequency $\omega(g)$ and the Fourier components Q_n and P_n can be expressed in terms of complete elliptic integrals, $Q(g,\varphi)$, $P(g,\varphi)$ are rational functions of the Jacobi elliptic functions. It is important in what follows that the solution of (4) in the form $q(t)=Q(g,\varphi(t))$, $p(t)=P(g,\varphi(t))$ holds true for both real and complex values of phase φ[2]. When $\operatorname{Im}\varphi\neq 0$ it describes closed trajectories in the complex phase space q,p. Some examples of the projections of these trajectories on the plane $\operatorname{Re}q$, $\operatorname{Im}q$ with the fixed g but different $\operatorname{Im}\varphi$ are shown in Fig. 2.

The relaxation effects cause trajectories (15) to be no longer closed. It is obvious that when the damping is small the trajectories are spirals with small pitch. Certain turns of the spiral come close to the trajectories (23). To describe

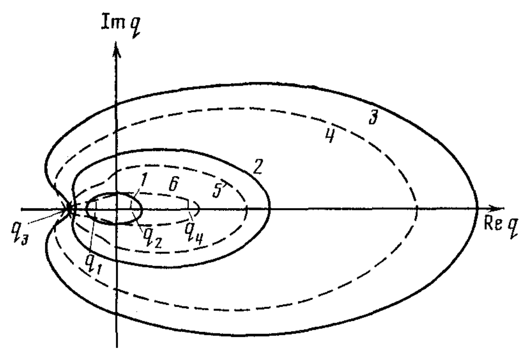

FIG. 2. Projection of the trajectories (4) on the plane Re q, Im q in the region of quasi-energy g, where there are two types of classically allowed motion: between points q_1,q_2 and q_3,q_4. Curves 1–6 correspond to increasing values of the phase imaginary part Im $\varphi(0<\operatorname{Im}\varphi<\Phi$, where $2i\Phi$ is the imaginary period of the functions $Q(g,\varphi)$, $P(g,\varphi)$. When Im $\varphi=0$ the trajectory corresponds to the segment (q_1,q_2). At some value of Im $\varphi=\theta$ the trajectory goes off to infinity, while at Im $\varphi=\Phi$ it coincides with the segment (q_3,q_4). The trajectories with $\Phi>\operatorname{Im}\varphi>0$ are shown dashed.

such motion in the interval $\Delta t \sim \Omega \gg 1$, it is convenient, in accordance with the idea of the averaging method (see Ref. 21), to switch from the oscillating functions $q_{1,2}(t)$, $p_{1,2}(t)$ to the functions $g_{1,2}(t)$, $\varphi_{1,2}(t)$:

$$q_j(t)=Q(g_j(t),\ \varphi_j(t)),\ p_j(t)=P(g_j(t),\ \varphi_j(t))\ \ (j=1,2). \tag{24}$$

Equations (15) for $g_{1,2}(t)$, $\varphi_{1,2}(t)$ and with Eq. (6) and (23) taken into consideration, take the form

$$\begin{aligned} &\dot{g}_j=-\Omega^{-1}\omega(g_j)\,\partial R/\partial\varphi_j,\\ &\dot{\varphi}_j=\omega(g_j)\,[(-1)^{j+1}+\Omega^{-1}(\partial R/\partial g_j)]\quad (j=1,2),\\ &g_1-g_2=-\Omega^{-1}R. \end{aligned} \tag{25}$$

The function R, which describes the relaxation of an oscillator, is periodic in φ_1 and φ_2

$$R\equiv R(g_1,g_2,\varphi_1,\varphi_2)=\sum_{n,m} R_{nm}(g_1,g_2)\exp[i(n\varphi_1+m\varphi_2)]. \tag{26}$$

The Fourier components $R_{nm}(g_1,g_2)$ are determined from (7), (23), and (24). Since, according to Eq. (5), the difference g_1-g_2 is small, it follows, that the terms with $m=n$ in Eq. (26) of the lowest order in Ω^{-1}, are slowly changing, while the terms with $m\neq n$ are rapidly oscillating [with the period $\sim\omega^{-1}_{(g_1)}\approx\omega^{-1}_{(g_2)}$]. Correspondingly, $g_{1,2}$ and $\varphi_{1,2}(t)$ are sums of the smooth $(\bar{g}_{1,2}(t),\bar{\varphi}_{1,2}(t))$ and rapidly oscillating terms, and the latter are small $(\sim\Omega^{-1})$.

The equations for slowly changing variables $\bar{g}_j,\bar{\varphi}_j$ look like equations (25) for g_j and φ_j, where R is given by Eq. (26), where only the diagonal terms (with $m=n$) are taken into account. The function $\bar{R}$ depends only on the half-sum of the phases $\bar{\varphi}=\frac{1}{2}(\bar{\varphi}_1+\bar{\varphi}_2)$. Besides, recognizing that $|\bar{g}_1-\bar{g}_2|\sim\Omega^{-1}$, the values of $\bar{g}_1$ and $\bar{g}_2$ in $R_{nn}(\bar{g}_1,\bar{g}_2)$ can be assumed equal. It can be proven that the system of equations for $\bar{g}(t)=\bar{g}_1(t)=\bar{g}_2(t)$, $\bar{\varphi}(t)$ has a simple integral

$$\omega^{-1}(\bar{g})\sum_n R_{nn}(\bar{g},\bar{g})\exp(2in\bar{\varphi})=C.$$

The value of the constant C is determined by the initial conditions (16) and (17) along the trajectory. Solving equations (18), it can be shown, that $C=0$ in the lowest order in Ω^{-1}, independently of $q_1(0)$ and $q_2(0)$, so that on the extremal trajectory

$$\sum_n R_{nn}(\bar{g},\bar{g})\,e^{2in\bar{\varphi}}=0,\quad \bar{g}\equiv\bar{g}(t)=\bar{g}_1(t)=\bar{g}_2(t),\quad \bar{\varphi}={}^1/{}_2[\bar{\varphi}_1(t)+\bar{\varphi}_2(t)], \tag{27}$$

or, taking into account the explicit form of the relaxation operator (7)

$$\begin{aligned} &(2\bar{n}+1)\sum_n(|Q_n|^2+|P_n|^2)[1-\exp(2in\bar{\varphi})]\\ &\quad+2\sum_n\operatorname{Im}(P_nQ_n)\exp(2in\bar{\varphi})=0,\\ &Q_n\equiv Q_n(\bar{g}),\quad P_n\equiv P_n(\bar{g}). \end{aligned} \tag{28}$$

Equation (27) allows us to express the slowly changing part of the half-sum of the phases $\bar{\varphi}$ in terms of the slowly changing part of quasi-energy $\bar{g}$, after which the equation for $\bar{g}(t)$

$$\dot{\bar{g}}=-i\Omega^{-1}\omega(\bar{g})\sum_n nR_{nn}(\bar{g},\bar{g})\exp(2in\bar{\varphi}),\quad \bar{\varphi}\equiv\bar{\varphi}(\bar{g}) \tag{29}$$

becomes closed. It also follows from Eqs. (25) and (27) that along the extremal trajectory

$$\bar{\varphi}_1(t)-\bar{\varphi}_2(t)=\bar{\varphi}_1(0)-\bar{\varphi}_2(0)+2\int_0^t dt_1\,\omega(\bar{g}(t_1)). \tag{30}$$

The values of $\bar{\varphi}_1(0)$, $\bar{\varphi}_2(0)$ and $\bar{g}(0)$ are determined by the values of $q_1(0)$ and $q_2(0)$ in Eq. (16). At the same time for the symmetric trajectories (20) we have

$$\operatorname{Im}\bar{\varphi}_1(0)=\operatorname{Im}\bar{\varphi}_2(0)$$

and on the whole

$$\operatorname{Im}\varphi_1(t)=\operatorname{Im}\varphi_2(t)$$

(we note, that this approximation is true in the limited region $|\operatorname{Im}[\varphi_1(0)-\varphi_2(0)]|$).

Equation (28) has an obvious solution $\bar{\varphi}=n\pi$ $(n=0,\mp1,\ldots)$. When $\bar{\varphi}=n\pi$ Eq. (29) goes over into the equation for the slowly changing part of quasi-energy on a classical trajectory (a) that describe the approach of an oscillator to a focus. However, along with this solution, Eq. (28) has another solution, one with $\operatorname{Im}\bar{\varphi}\neq 0$. Equations (27)–(29) are ture, when the latter does not pass through the singularity where $|Q(\bar{g},\bar{\varphi})|\to\infty$, $|P(\bar{g},\bar{\varphi})|\to\infty$.

4. TRANSITIONS PROBABILITIES FOR THE CASE OF WEAK DAMPING

The probability to leave a stable state k $(k=1,2)$ is determined by the action on the trajectory (20), which goes according to Eqs. (16) and (16a) out of the vicinity of the focus f_k into the vicinity of a saddle point S. This trajectory is described by the real solution of Eq. (28) for $\exp(2i\bar{\varphi})$, which is not equal to unity (i.e., $\operatorname{Im}\bar{\varphi}\neq 0$, $\operatorname{Re}\bar{\varphi}=\pi n$). The sign of $\operatorname{Im}\bar{\varphi}$ along the entire trajectory (vanishing of $\operatorname{Im}\bar{\varphi}$ signifies either tangency or intersection with the trajectory corresponding to the solution $\operatorname{Im}\bar{\varphi}=0$, which is possible only in the singular points, the saddle and the foci). As follows from Eqs. (17), (18), and (24), the sign of $\dot{g}$ does not change either, i.e., quasi-energy changes monotonically along the trajectory. To find the explicit form of the values of $\bar{\varphi}$ and $\dot{\bar{g}}$ in the vicinity of foci is not difficult, by starting from Eqs. (17), (18), and (24) (we note here that in the lowest order in Ω^{-1} these equations have the solution $g_1(0)=g_2(0)$ for any $q_1(0),q_2(0)$ which are close to q_f); the sign of $\dot{\bar{g}}$ turns out to indicate motion towards the saddle.

To calculate the action along the trajectories described by Eqs. (24) and (27)–(30), it is convenient to use the relation

$$\begin{aligned} &p_j\dot{q}_j=-\omega^{-1}(g_j)\,\varphi_j\dot{g}_j+\frac{d}{dt}S_0(q_j,g_j)\qquad (j=1,2),\\ &S_0(Q(g,\varphi),g)=\int_0^\varphi d\varphi' P(g,\varphi')\,\frac{\partial Q(g,\varphi')}{\partial\varphi'}, \end{aligned} \tag{31}$$

which stems from Eqs. (23) and (24); here $S_0(q,g)$ is the simplified action for periodic trajectories (4) and (23) (cf. Ref. 6). Taking Eq. (31) into account, accuracy to terms $\sim\Omega^{-1}$, the action $S(t)$ [Eq. (15)] on the trajectory coming from the small vicinity of the focus f is equal to

$$S(t)=-2\int_{g_f}^{\bar g(t)} d\bar g\,\omega^{-1}(\bar g)\bar\varphi(\bar g)+S_0(q_1(t),\bar g(t))+S_0(q_2(t),\bar g(t)),$$
$$q_j(t)=Q(\bar g(t),\ \varphi_j(t))\quad (j=1,2). \tag{32}$$

Starting from the Eq. (32), we obtain the value of G (13), which determines the probability of transition from the focus f and equals the imaginary part of the action along the trajectory (20) going from the focus to the saddle

$$G=-2\int_{g_f}^{g_s} dg\,\omega^{-1}(g)\,\mathrm{Im}\,\bar\varphi(g),\qquad \Omega^{-1}<\lambda\ll 1 \tag{33}$$

(here g_f and g_s are the values of quasi-energy g in the focus and in the saddle point).

Equations (13), (28), and (33) reduce the problem of calculation of $\ln W$ in the limiting case of small damping to the problem of solving Eq. (28) for $\bar\varphi(g)$ and subsequent integration. The value of G in Eq. (33) depends only on two dimesionless parameters, β and $\bar n$, i.e., on the field intensity made dimensionless, and on the temperature, while the damping parameter Γ does not enter in (33). In this respect, the result of Eqs. (13) and (33) for $\ln W$ is similar to the expression for the logarithmic probability of an activation transition through a potential barrier, when interaction with the medium is weak $\ln W_a \approx -\Delta U/T$. The quasi-energy $|g_s-g_f|$ difference assumes the role of the barrier height ΔU in Eqs. (13) and (33), and the quantity $-2\lambda^{-1}\omega^{-1}(g)\,\mathrm{Im}\,\bar\varphi(g)$ plays the role of the reciprocal temperature (which in this case depends on quasi-energy g). The quantity $-2\lambda^{-1}\omega^{-1}(g)\,\mathrm{Im}\,\bar\varphi(g)$ plays the same role in the expression for the quasistationary distribution in quasi-energy in the vicinity of the initially occupied stable state. This distribution follows from Eqs. (10) and (32). It can be found also in an alternative way, by expressing kinetic equation (6) in matrix form with the eigenfunctions $\Psi_n(q)$ of the operator g in Eq. (5), localized in the vicinity of the stable state. The quasiclassical solution of the corresponding equation for the element of the matrix ρ_{nm} for $\Omega^{-1}\ll\lambda\ll 1$ is

$$\rho_{nm}=\delta_{nm}(\rho_0)_{nn}\exp\left[2\lambda^{-1}\int_{g_f}^{g_n} dg\,\omega^{-1}(g)\,\mathrm{Im}\,\varphi(g)\right],$$

which is in agreement with the result of Eqs. (10) and (32).

The results of the numerical calculations of G_1 and G_2, which determine the probabilities for transitions from the foci f_1 and f_2 (corresponding to the smaller and larger amplitudes of the forced oscillations), are depicted in Fig. 3. The value of G_1 is monotonically diminishing, while the value of G_2 is monotonically increasing, with increase in the effective field β. At the bifurcation points, where the focus f_k $(k=1,2)$ merges with the saddle point, G_k turns into zero. In the area of small $\bar n$ the values G_1 (for small β) and G_2 (for relatively large β) have a strong dependence on $\bar n$ when $n<\beta\ll 1$ G_1 experiences a logarithmic increase with decrease of β. With increase of temperature, $G_{1,2}$ decrease. When $\bar n>1$ the $(\bar n+\frac{1}{2})G_{1,2}$ curves cease to depend on $\bar n$, with accuracy of several percents, and go over into the curves of Fig. 2

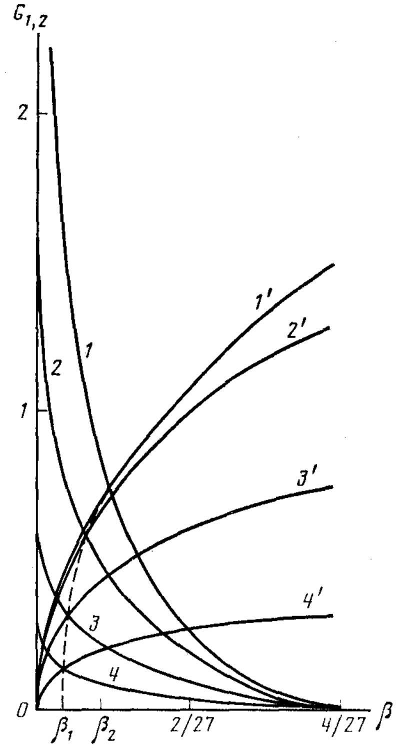

FIG. 3. Dependence of G_1 and G_2 (the curves 1–4 and 1′, 4′, respectively) on β in the limiting case of weak damping for various $\bar n$: $\bar n = 0, 0.01, 0.2, 1$ $G_1\to\infty$ on the curve 1 as $\beta\to 0$; the form of this curve in the area of $\beta\lesssim 0.005$ is well described by the formula (41). The line of the kinetic "phase transition" $\beta=\beta_0$ on which $G_1=G_2$ is shown dashed. The points $\beta_1\simeq 0.013$ and $\beta_2\simeq 0.036$ are the values of β_0 for $\bar n\gg 1^3$ and $\bar n=0$, respectively.

in Ref. 3, plotted without considering quantum effects.

Explicit expressions for $G_{1,2}$ can be found in several limiting cases. When the temperatures are relatively high, $\bar n\gg 1$, it follows from Eq. (28), that $|\mathrm{Im}\bar\varphi|\ll 1$. Retaining only the non-zero terms of the lowest order in $\mathrm{Im}(\varphi)$ in Eq. (28) and taking Eq. (23) into account, we obtain

$$\mathrm{Im}\,\bar\varphi(g)=(2\bar n+1)^{-1}m_2(g)/m_1(g),\quad \bar n\gg 1,$$
$$m_1(g)=\int_0^{2\pi} d\varphi\left[\left(\frac{\partial Q}{\partial\varphi}\right)^2+\left(\frac{\partial P}{\partial\varphi}\right)^2\right],$$
$$m_2(g)=\int_0^{2\pi} d\varphi\left(Q\frac{\partial P}{\partial\varphi}-P\frac{\partial Q}{\partial\varphi}\right),$$
$$Q\equiv Q(g,\varphi),\qquad P\equiv P(g,\varphi) \tag{34}$$

(the asymptotic behavior of Eq. (34) provides good description of $G_{1,2}$ in the entire region $\bar n>1$). It follows from (33) and (34), that $G\propto T^{-1}$ in the classical limit $T\gg\hbar\omega_F$. The quantity $\lambda^{-1}TG$ plays the part of the transition activation energy. It depends on the single parameter β. Equations (13), (33), and (34) for $\ln W$ coincide completely with the results of Ref. 3.

It is easy to solve Equation (28) in the vicinity of the point of bifurcation $\beta=\beta_B$ when the initially occupied focus f is close to the saddle point. The characteristic frequency of the oscillations is small in the vicinity of such a focus,

$\omega(g_f) \ll 1$ (at the same time we assume, that it exceeds the damping $\omega(g_f) \gg \Omega^{-1}$). Then $|\mathrm{Im}\varphi(g)| \sim \omega(g) \ll 1$ and is also given by Eq. (34), and for arbitrary values of $\bar{n}$ too. It is possible to explicitly calculate the ratio of the integrals $m_2(g)/m_1(g)$, appearing in Eq. (34), in the vicinity of the bifurcation points. The obtained result is

$$G_1 \approx 9(2\bar{n}+1)^{-1}(4/27-\beta)^{5/4},\ \Omega^{-2} \ll 4/27-\beta \ll 1,\ \beta_B^{(2)} \approx 4/27, \tag{35}$$

$$G_2 \approx 4(2\bar{n}+1)^{-1}\beta^{1/2},\ \Omega^{-2} \ll \beta \ll 1,\ \beta_B^{(1)} \approx 0.$$

The values of G_1, G_2 decrease quickly (non-analytically in $|\beta-\beta_B|$) as the bifurcation point is approached. We note that the probability for tunneling from state 1 near the bifurcation point is much smaller $|\ln W_t| \propto (4/27-\beta)^{5/4}$ (Ref. 13).

Using the proposed approach, it is not difficult to account for dissipation when calculating the probability of transitions, a problem which has attracted much attention lately (cf. Ref. 22 and 23). It is easy to see that the dissipation-induced correction to the action in Eq. (32) along the extremal trajectory is $\sim\Omega^{-1}$. In the region of strong quantum damping, when $\Omega^{-1} \gg \lambda$ [see Eq. (22)], this correction leads to an exponentially large factor in W, even if the damping is classically weak $\Omega^{-1} \ll 1$.

5. PROBABILITY OF LEAVING A STABLE STATE THAT CORRESPONDS TO A SMALLER AMPLITUDE OF FORCED OSCILLATIONS IN A WEAK FIELD

The probability to leave state 1 and the form of distribution in the vicinity f_1 in relatively weak fields $\beta \ll 1$ are of considerable interest for many concrete physical systems, where the condition $\beta \ll 1$ is often satisfied (cf. Ref. 7). Using the results obtained in Sec. 2, they can be explicitly found for the case of weak damping $\Omega^{-1} \ll 1$. They turn out to depend in a complex way (non-analytically for the limiting case $\Omega^{-1} \to 0$) on Planck's number $\bar{n}$ in the relaxation model, Eq. (7). This follows from the fact, that $\mathrm{Im}\,\bar{\varphi}(g)$ defined by Eq. (28) approaches as $\bar{n} \to 0$ the value $\theta(g)$ at which the extremal trajectory tends to infinity. $|Q(g, i\theta(g))|^{-1} = 0$. When $|\mathrm{Im}\,\bar{\varphi}(g)| \gg |\theta(g)|$ the averaging method in the form of Eqs. (24) and (27)–(30) is not applicable. It can be shown, though, that when $\bar{n} \gg \Omega^{-3}$, the difference $|\mathrm{Im}\bar{\varphi}(g) - \theta(g)|$ is not too small and formulas (27)–(30) hold true.

When β is small, the expressions for $\omega(g)$ and $\theta(g)$, and also for $Q_n(g)$ and $P_n(g)$, are simplified. In particular,

$$\omega(g) \approx 2g^{1/2},\quad \theta(g) = {}^1\!/_2 \ln\,[64\beta^{-1}g^2(1-2g^{1/2})^{-1}],$$
$$1/4 - g \gg \beta,\quad g \gg \beta^{1/2}, \tag{36}$$
$$g_{f_1} \approx 1/4,\ g_s \approx \beta^{1/2}\ (\beta \ll 1).$$

Substituting the corresponding expressions into Eq. (28), we find that for not very low temperatures

$$\mathrm{Im}\,\bar{\varphi}(g) \approx {}^1\!/_2 \ln\,[(\bar{n}+1)/\bar{n}] = {}^1\!/_2 \hbar\omega_F/T,\quad g \gg \beta^{1/2},\quad \bar{n} \gg \beta,\quad \beta \ll 1;$$
$$G_1 = \hbar\omega_F/2T - \delta G_1,\ \delta G_1 \approx \xi\beta^{1/4}. \tag{37}$$

When $\bar{n} \sim \beta \ll 1$ Eq. (37) coincides with the limit $\beta \ll 1$ in (33) and (34) (in that case $\xi \simeq 0.98 \cdot \lambda \bar{n}^{-1}$, cf. Ref. 3). Equation (37), however, is also true for $\bar{n} < 1$ ($\xi \simeq 1.86\bar{n}^{-1/4} \gg 1$ when $\bar{n} \ll 1$). As one can see from Eq. (37), the transition probability and the form of distribution of quasi-energy are described when $\bar{n} \gg \beta$ by a simple activation law, while the main term in G_1 does not depend on the field. When $\bar{n} \sim \beta \ll 1$ the value of $\mathrm{Im}\bar{\varphi}(g) - \theta(g)$ is determined by the ratio $\bar{n}/\beta$. Equation (28) in that region can be reduced to the form

$$(1-z)^3 - z(1+z)\bar{n}\exp\,[2\theta(g)] = 0,$$
$$z \equiv z(g) = \exp\{2\,[\mathrm{Im}\,\bar{\varphi}(g) - \theta(g)]\},$$
$$\bar{n} \ll 1,\ \beta \ll 1\ (z < 1). \tag{38}$$

The solution to Eq. (38), with (36) taken into account, can be easily tabulated for various ratio $\bar{n}/\beta$. When $\bar{n} \gg \beta$ it coincides with (37). In the reverse limiting case

$$\mathrm{Im}\,\bar{\varphi}(g) = \theta(g) - {}^1\!/_2\{2\bar{n}\exp\,[2\theta(g)]\}^{1/3}\ ({}^1\!/_4 - g \gg \beta),$$
$$G_1 = {}^1\!/_2 \ln(4\beta^{-1}) - {}^3\!/_2 - (4\pi/3^{5/2})(\bar{n}/\beta)^{1/3},\ \bar{n} \ll \beta. \tag{39}$$

The main terms in (39) have a logarithmic dependence on the field intensity $F^2 \propto \beta$, i.e., the distribution with respect to the quasi-energy and the probability W_{12} of the $1 \to 2$ transition have a power-law dependence on F^2. In particular,

$$W_{12} \propto \beta^{1/2\lambda} \propto F^{1/\lambda},\ \lambda = |3\hbar\gamma/8\omega_F^2(\omega_F - \omega_0)|\quad (\Omega^{-3} \ll \bar{n} \ll \beta). \tag{40}$$

In Ref. 13, 15, and 17 the power-law dependence on F was also obtained in a relatively weak field, when calculating the probability W_t of tunnelings from the focus f_1, $W_t \propto F^2/\lambda$. Obviously, the exponent in Eq. (40) is one half less than the exponent in W_t. Thus, the "above-the-barrier" transitions induced by dissipation turn out to be many times less probable than the tunnelings. G_1 rapidly decreases with the increase in temperature (but W_{12} increases), and the dependence of G_1 on $\bar{n}$ within the region $\bar{n} \ll \beta$ is not analytical.

It is evident from Eq. (39) that $\bar{n} \to 0$ and $\mathrm{Im}\bar{\varphi}(g) \to \theta(g)$ with decrease in temperature. At the same time, the extremal trajectory goes off to infinity, and the proposed approach is inapplicable. Still, analysis shows that when $\bar{n} \ll \Omega^{-4}$ the transition probability in the zeroth approximation in $\bar{n}$ and Ω^{-1} is formally given by Eqs. (13), (28), and (33) with $\bar{n} = 0$ (even though it turns out that $\mathrm{Im}\bar{\varphi}(g) > \theta(g)$ in a definite region of g; we note that it is possible to find the explicit solution of Eq. (28) for arbitrary β). The corresponding results have been used to plot curve 1 for G_1 in Fig. 3. When β is small we have

$$G_1 = \ln\beta^{-1} - 3,\ \bar{n} \ll \Omega^{-4} \ll \beta^2 \ll 1. \tag{41}$$

The main term $\ln\beta^{-1}$ in Eq. (41) is two times larger than the main term in G_1 of Eq. (39). This means that the transition probability sharply changes its value, when $\bar{n}$ changes slightly $\sim\Omega^{-3}$. The term $\ln\beta^{-1}$ coincides with the main term in $\lambda\,|\ln W_t|$. At the same time, the large term in $\ln W_{12}$, which is independent of the field and equals $3\lambda^{-1}$, is larger by $\lambda^{-1} \cdot \ln 2$ than the one in $\ln W_t$ [13]. This confirms the conclusion that even in a weak field and as $\bar{n} \to 0$ the departure from state 1 is not by tunneling (tunneling from state 2 is impossible, regardless of the values of β).

CONCLUSION

The existence of diffusion in quasi-energy, which is due to interaction with the medium (and accompanies a drift in the direction of a stable state), leads to the conclusion that both stable states of an oscillator are strictly speaking metastable at any temperature. The probabilities W_{12} and W_{21} of transitions $1\to 2$ and $2\to 1$ are finite. At the same time the values of W_{12} and W_{21}, differ strongly exponentially for all values of the parameters. This pertains also to the stationary populations $\rho^{(1)}$ and $\rho^{(2)}$ of the states 1 and 2:

$$\rho^{(1)}/\rho^{(2)}=W_{21}/W_{12}=\text{const}\exp[(G_1-G_2)/\lambda]. \qquad (42)$$

Only at a certain relation between the parameters β, Ω, and $\bar{n}$ i.e., between the field intensity F^2, the frequency detuning $\omega_F-\omega_0$, the damping Γ, and the temperature, when $|G_1-G_2|\lesssim\lambda$, does a diffuse kinetic first-order "phase transition" occur and the populations $\rho^{(1)}$ and $\rho^{(2)}$ are close order (cf. Ref. 3).

In the classical theory, the position of the phase-transition line $\beta_0(\Omega)$ on the β,Ω plane did not depend on T (Ref. 3). It follows from the discussed result that this holds true up to $\bar{n}\approx 1$. A phase transition at lower T can be caused by a change in temperature. As can be seen from Fig. 3, with the decrease in temperature G_1 grows faster than G_2, and as a result, the region (β,Ω) expands; it is in this region where the state I is mostly occupied (it corresponds to a smaller amplitude of the forced oscillations of the oscillator).

The results of the calculations of $G_{1,2}$ for the case of weak damping were given above $\Gamma\ll|\omega_F-\omega_0|$. When $\Gamma\sim|\omega_F-\omega_0|$ the value of G_k can be found analytically near the bifurcation points $\beta=\beta_B^{(3-k)}(\Omega)$ where the stable stationary state k ($k=1,2$) merges with the unstable state. The results here coincide with the results obtained in the classical theory (see Refs. 3 and 12), if one replaces T in the expression for $\ln W_{kk'}$ by $\hbar\omega_F/\bar{n}+\frac{1}{2}$.

The fact that the obtained transition probability significantly exceeds its value obtained by accounting for tunneling decay only (which is possible for state 1) is important from the point of view of the analysis of a problem of collisionless dissociation of molecules in a laser field.[7] The difference in probabilities is especially significant at not very low temperatures $\bar{n}\gg(\Gamma/|\omega_F-\omega_0|)^3$ for weak fields [cf. Eqs. (39) and (40)]. It can be shown that even a relatively weak interaction with the medium, modulating oscillator's frequency, or small random variations of the field intensity in time (with the characteristic frequencies $\Delta w_F\gg|\omega_F-\omega_0|$) can lead to a strong increase in W_{12} in weak fields, and at lower T. The authors are grateful to A. S. Ioselevich and M. A. Krivoglaz for the discussion of the results of the paper, and to E. V. Mozdor for carrying out the numerical calculations.

[1)] In quantum theory it turns out to be more convenient to calculate the density matrix in the coordinate rather than in Wigner representation, especially in the presence of weak damping.

[2)] It follows from the theory of elliptic functions (see for example Ref. 20), that the functions $Q(g,\varphi)$ and $P(g,\varphi)$ are periodic in Re φ and Im φ. They have two poles of the second order in the rectangle of the periods. Later in the text we will assume that Im $\varphi=0$ for the real trajectories passing through the region of attraction of the initially occupied stable state.

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