1. [20 pts] In order to derive the properties of the spherical harmonics, we need to determine the action of the angular momentum operator in spherical coordinates. Just as we have \( \langle x | P_x | \psi \rangle = -i \hbar \frac{d}{dx} \langle x | \psi \rangle \), we should find a similar expression for \( \langle r \theta \phi | \vec{L} | \psi \rangle \). From \( \vec{L} = \vec{R} \times \vec{P} \) and our knowledge of momentum operators, it follows that
\[
\langle r \theta \phi | \vec{L} | \psi \rangle = -i \hbar \left( \vec{e}_x \left( y \frac{d}{dz} - z \frac{d}{dy} \right) + \vec{e}_y \left( z \frac{d}{dx} - x \frac{d}{dz} \right) + \vec{e}_z \left( x \frac{d}{dy} - y \frac{d}{dx} \right) \right) \langle r \theta \phi | \psi \rangle.
\]
Cartesian coordinates are related to spherical coordinates via the transformations
\[
x = r \sin \theta \cos \phi \\
y = r \sin \theta \sin \phi \\
z = r \cos \theta
\]
and the inverse transformations
\[
r = \sqrt{x^2 + y^2 + z^2} \\
\theta = \arctan \left( \frac{\sqrt{x^2 + y^2}}{z} \right) \\
\phi = \arctan \left( \frac{y}{x} \right).
\]
Their derivatives can be related via expansions such as
\[
\partial_x = \frac{\partial r}{\partial x} \partial_r + \frac{\partial \theta}{\partial x} \partial_\theta + \frac{\partial \phi}{\partial x} \partial_\phi.
\]
Using these relations, and similar expressions for \( \partial_y \) and \( \partial_z \), find expressions for \( \langle r \theta \phi | L_x | \psi \rangle \), \( \langle r \theta \phi | L_y | \psi \rangle \), and \( \langle r \theta \phi | L_z | \psi \rangle \), involving only spherical coordinates and their derivatives.
\[
\begin{align*}
\partial_x r &= \frac{\partial x}{r} = \sin \theta \cos \phi \\
\partial_x \theta &= \frac{\partial x}{\sqrt{x^2 + y^2}} = \frac{\cos \theta \cos \phi}{r} \\
\partial_x \phi &= \frac{\partial x}{x^2 + y^2} = -\frac{\csc \theta \sin \phi}{r} \\
\text{So } \frac{d}{dx} &= \sin \theta \cos \phi \partial_r + \frac{\cos \theta \cos \phi}{r} \partial_\theta - \frac{\csc \theta \sin \phi}{r} \partial_\phi \\
\partial_y r &= \frac{\partial y}{r} = \sin \theta \sin \phi \\
\partial_y \theta &= \frac{\partial y}{\sqrt{x^2 + y^2}} = \frac{\cos \theta \sin \phi}{r} \\
\partial_y \phi &= \frac{\partial y}{x^2 + y^2} = \frac{\csc \theta \cos \phi}{r} \\
\text{So } \frac{d}{dy} &= \sin \theta \sin \phi \partial_r + \frac{\cos \theta \sin \phi}{r} \partial_\theta + \frac{\csc \theta \cos \phi}{r} \partial_\phi \\
\partial_z r &= \frac{\partial z}{r} = \cos \theta \\
\partial_z \theta &= \frac{\partial z}{\sqrt{x^2 + y^2}} = \frac{\csc \theta \cos \phi}{r} \\
\partial_z \phi &= \frac{\partial z}{x^2 + y^2} = \frac{\csc \theta \cos \phi}{r} \\
\text{So } \frac{d}{dz} &= \cos \theta \partial_r + \frac{\csc \theta \cos \phi}{r} \partial_\theta + \frac{\csc \theta \cos \phi}{r} \partial_\phi.
\end{align*}
\]
\[ \begin{aligned} 
\partial_z \theta &= -\frac{z^2}{r^2} \sqrt{\frac{x^2+y^2}{z^2}} = -\frac{\sin \theta}{r} \\
\partial_z \phi &= 0 \\
\text{So } \frac{d}{dz} &= \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta 
\end{aligned} \]

Now

\[ \langle r \theta \phi | L_x | \psi \rangle = -i\hbar \left( y \frac{d}{dz} - z \frac{d}{dy} \right) \langle r \theta \phi | \psi \rangle \]

So we can say

\[ L_x = i\hbar \left( y \frac{d}{dz} - z \frac{d}{dy} \right) \]

\[ = -i\hbar \left( r \sin \theta \cos \theta \sin \phi \partial_r - \sin^2 \theta \sin \phi \partial_\theta - r \sin \theta \cos \theta \partial_r - \cos^2 \theta \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi \right) \]

Which means

\[ \langle r \theta \phi | L_x | \psi \rangle = -i\hbar \left( -\sin \phi \partial_\theta - \cot \theta \cos \phi \partial_\phi \right) \langle r \theta \phi | \psi \rangle \]

Similarly we can say

\[ L_y = -i\hbar \left( \frac{d}{dx} - \frac{d}{dz} \right) \]

\[ = -i\hbar \left( r \sin \theta \cos \theta \cos \phi \partial_r + \cos^2 \theta \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\theta - r \sin \theta \cos \theta \cos \phi \partial_r + \sin^2 \theta \cos \phi \partial_\theta \right) \]

so that

\[ \langle r \theta \phi | L_y | \psi \rangle = -i\hbar \left( \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi \right) \langle r \theta \phi | \psi \rangle \]

Lastly, we have

\[ L_z = -i\hbar \left( \frac{d}{dy} - \frac{d}{dx} \right) \]

\[ = -i\hbar \left( r \sin^2 \theta \sin \phi \cos \phi \partial_r + \sin \theta \cos \theta \sin \phi \cos \phi \partial_\theta + \cos^2 \phi \partial_\phi \\
+ r \sin^2 \theta \sin \phi \cos \phi \partial_r - \sin \theta \cos \theta \sin \phi \cos \phi \partial_\theta + \sin^2 \phi \partial_\phi \right) \]

so that

\[ \langle r \theta \phi | L_z | \psi \rangle = -i\hbar \partial_\phi \langle r \theta \phi | \psi \rangle \]
2. [15pts] From your previous answer and the definition $L^2 = L_x^2 + L_y^2 + L_z^2$, prove that

$$\langle r\theta\phi|L^2|\psi\rangle = -\hbar^2 \left( \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial^2\phi} \right) \langle r\theta\phi|\psi\rangle.$$

$$L^2 = -\hbar^2 \left[ (\sin\phi \partial_\theta + \cot\theta \cos\phi \partial_\phi)(\sin\phi \partial_\theta + \cot\theta \cos\phi \partial_\phi) 
+ (\cos\phi \partial_\theta - \cot\theta \sin\phi \partial_\phi)(\cos\phi \partial_\theta - \cot\theta \sin\phi \partial_\phi) + \partial_\phi^2 \right]$$

$$= -\hbar^2 \left[ \sin^2\phi \partial_\theta^2 + \cot\theta \sin\phi \cos\phi \partial_\theta \partial_\phi - \csc^2\theta \sin\phi \cos\phi \partial_\theta + \cot\theta \sin\phi \cos\phi \partial_\theta \partial_\phi + \cot^2\theta \cos^2\phi \partial_\phi^2 - \cot^2\theta \cos\phi \sin\phi \partial_\theta + \cos^2\phi \partial_\theta^2 - \cot \theta \cos\phi \sin\phi \partial_\phi \partial_\theta + \cot \theta \sin^2\phi \partial_\theta \partial_\phi + \cot^2 \theta \sin^2\theta \partial_\phi^2 + \cot^2 \theta \sin\phi \cos\phi \partial_\theta + \partial_\phi^2 \right]$$

$$= -\hbar^2 \left[ \partial_\theta^2 + \cot\theta \partial_\theta + (1 + \cot^2\theta) \partial_\phi^2 \right]$$

Noting that

$$\frac{1}{\sin\theta} \partial_\theta \sin\theta \partial_\theta = \partial_\theta^2 + \cot\theta \partial_\theta$$

and

$$1 + \cot^2\theta = \frac{1}{\sin^2\theta}$$

the proof is complete.
3. [10 pts] We can factorize the Hilbert space of a 3-D particle into radial and angular Hilbert spaces, \( \mathcal{H}^{(3)} = \mathcal{H}^{(r)} \otimes \mathcal{H}^{(\Omega)} \). Two alternate basis sets that both span \( \mathcal{H}^{(\Omega)} \) are \( \{ |\theta \phi \rangle \} \) and \( \{ |\ell m \rangle \} \). As the angular momentum operator lives entirely in \( \mathcal{H}^{(\Omega)} \), we can use our results from problem 11.1 to derive an expression for \( \langle \theta \phi | L_z | \ell m \rangle \). Combine this with the formula \( L_z | \ell m \rangle = \hbar m | \ell m \rangle \), to derive and then solve a differential equation for the \( \phi \)-dependence of \( \langle \theta \phi | \ell m \rangle \). Your solution should give \( \langle \theta \phi | \ell m \rangle \) in terms of the as of yet unspecified initial condition \( \langle \theta | \ell m \rangle \equiv \langle \theta, \phi | \ell m \rangle |_{\phi=0} \). What restrictions does this solution impose on the quantum number \( m \), which describes the \( z \)-component of the orbital angular momentum? Since \( m_{\text{max}} = \ell \), what restrictions are then placed on the total angular momentum quantum number \( \ell \)?

\[
\langle \theta \phi | L_z | \ell m \rangle = -i\hbar \frac{\partial}{\partial \phi} \langle \theta \phi | \ell m \rangle
\]

and

\[
\langle \theta \phi | L_z | \ell m \rangle = \hbar m \langle \theta \phi | \ell m \rangle
\]

Thus

\[-i\hbar \frac{\partial}{\partial \phi} \langle \theta \phi | \ell m \rangle = \hbar m \langle \theta \phi | \ell m \rangle\]

The solution to this simple first-order differential equation is

\[
\langle \theta \phi | \ell m \rangle = \langle \theta 0 | \ell m \rangle e^{im\phi}
\]

Since the wave-function must be single valued, we require \( m \) to be a whole integer. As \( m_{\text{max}} = \ell \), this implies that \( \ell \) must be a whole integer also.
4. [10 pts] Using $L_\pm = L_x \pm i L_y$ we can use the relation $L_+ |\ell, \ell\rangle = 0$ and the expressions from problem 11.1 to write a differential equation for $\langle \theta \phi |\ell, \ell\rangle$. Plug in your solution from 11.3 for the $\phi$-dependence, and show that the solution is $\langle \theta \phi |\ell, \ell\rangle = c_\ell e^{i \ell \phi} \sin^\ell \theta$. Determine the value of the normalization coefficient $c_\ell$ by performing the necessary integral.

We have $\langle \theta \phi |L_+|\ell, \ell\rangle = 0$
This implies $\langle \theta \phi |L_x|\ell, \ell\rangle + i \langle \theta \phi |L_y|\ell, \ell\rangle = 0$
Using the expressions from 11.1 gives $(- \sin \theta \partial_\theta - \cot \theta \cos \phi \partial_\phi + i \cos \phi \partial_\phi)\langle \theta \phi |\ell, \ell\rangle = 0$
This simplifies to $(i (\cos \phi + i \sin \phi) \partial_\theta - \cot \theta (\cos \phi + i \sin \phi) \partial_\phi)\langle \theta \phi |\ell, \ell\rangle = 0$
Factoring out the $e^{i \phi}$ gives $(i \partial_\theta - \cot \theta \partial_\phi)\langle \theta \phi |\ell, \ell\rangle = 0$
Plugging in the solution from 11.3 gives $(i \partial_\theta - i \ell \cot \theta)\langle \theta 0 |\ell, \ell\rangle e^{-i \ell \phi} = 0$
which reduces to $\partial_\theta \langle \theta 0 |\ell, \ell\rangle = \ell \cot \theta \langle \theta 0 |\ell, \ell\rangle$

Now $\partial_\theta \sin^\ell \theta = \ell \sin^{\ell-1} \theta \cos \theta = \ell \cot \theta \sin^\ell \theta$
So the solution is $\langle \theta \phi |\ell, \ell\rangle = c_\ell e^{i \ell \phi} \sin^\ell \theta$

The normalization integral is $\int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \langle |\theta \phi |\ell, \ell\rangle^2 = 1$
With our solution this becomes $|c_\ell|^2 \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \sin^{2\ell} \theta = 1$
Performing the phi integral gives $2\pi |c_\ell|^2 \int_0^\pi \sin \theta d\theta \sin^{2\ell} \theta = 1$
Substitution with $u = \cos \theta$ gives $2\pi |c_\ell|^2 \int_0^1 du (1 - u^2)^\ell = 1$
Since the integrand is even, this reduces to $4\pi |c_\ell|^2 \int_0^1 du (1 - u^2)^\ell = 1$
From Mathematica we get $2\pi |c_\ell|^2 \frac{\Gamma(\ell+1/2)\Gamma(\ell+1)}{\Gamma(\ell+3/2)\Gamma(\ell+1)} = 1$
which gives $c_\ell = \sqrt{\frac{\Gamma(\ell+3/2)}{2\pi \Gamma(\ell+1)^2}}$

Thus we have $\langle \theta \phi |\ell, \ell\rangle = \sqrt{\frac{\Gamma(\ell+3/2)}{2\pi \Gamma(\ell+1)^2}} \sin^\ell \theta e^{i \ell \phi}$

For the special case $\ell = 3$ this gives $\langle \theta \phi |33\rangle = \frac{1}{8} e^{3i \phi} \sqrt{\frac{35}{\pi}} \sin^3 \theta$, which agrees with the spherical harmonic $Y_3^3(\theta, \phi)$ up to a non-physical phase factor.
5. [10 pts] Using \( L_- |\ell m\rangle = \hbar \sqrt{\ell(\ell + 1) - m(m - 1)} |\ell, m - 1\rangle \) together with your previous answers to derive an expression for \( \langle \theta \phi | \ell, m - 1 \rangle \) in terms of \( \langle \theta \phi | \ell m \rangle \). Explain how in principle you can now recursively calculate the value of the spherical harmonic \( Y^m_\ell(\theta \phi) \equiv \langle \theta \phi | \ell m \rangle \) for any \( \theta \) and \( \phi \) and for any \( \ell \) and \( m \). Follow your procedure to derive properly normalized expressions for spherical harmonics for the case \( \ell = 1, m = -1, 0, 1 \).

To construct the other \( m \) states for the same \( \ell \), we can begin from the expression

\[
\langle \theta \phi | L_- |\ell m\rangle = \hbar \sqrt{\ell(\ell + 1)(\ell - m + 1)} \langle \theta \phi | \ell, m - 1 \rangle
\]

Now

\[
\langle \theta \phi | L_- |\ell m\rangle = \langle \theta \phi | L_x |\ell m\rangle - i \langle \theta \phi | L_y |\ell m\rangle
\]
\[
= -i \hbar (- \sin \phi \partial_\theta - \cot \theta \cos \phi \partial_\phi) \langle \theta \phi | \ell m \rangle - \hbar (\cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi) \langle \theta \phi | \ell m \rangle
\]
\[
= \hbar e^{-i\phi} (- \partial_\theta + i \cot \theta \partial_\phi) \langle \theta \phi | \ell m \rangle
\]

Putting the pieces together gives

\[
\langle \theta \phi | \ell, m - 1 \rangle = \frac{e^{-i\phi} (- \partial_\theta + i \cot \theta \partial_\phi)}{\sqrt{\ell + m} (\ell - m + 1)} \langle \theta \phi | \ell m \rangle
\]

Starting from our expression for \( \langle \theta \phi | \ell \ell \rangle \), we can find \( \langle \theta \phi | \ell, \ell - 1 \rangle \) by applying the above differential formula. Successive iterations will then generate all the remaining \( \langle \theta \phi | \ell m \rangle \) states.
6. [10 pts] A particle of mass $M$ is constrained to move on a spherical surface of radius $a$.

Does the system live in $\mathcal{H}^{(3)}$, $\mathcal{H}^{(r)}$, or $\mathcal{H}^{(\Omega)}$? What is the Hamiltonian? What are the energy levels and degeneracies? What are the wavefunctions of the energy eigenstates?

Because the radial motion is constrained to a fixed value, it is only necessary to consider the dynamics in $\mathcal{H}^{(\Omega)}$.

The Hamiltonian is then

$$H = \frac{L^2}{2Ma^2}$$

Choosing simultaneous eigenstates of $L^2$ and $L_z$, we have

$$H|\ell,m\rangle = \frac{\hbar^2\ell(\ell + 1)}{2Ma^2}|\ell,m\rangle$$

so that

$$E_\ell = \frac{\hbar^2\ell(\ell + 1)}{2Ma^2}$$

and

$$d_\ell = 2\ell + 1$$

The wavefunctions are the spherical harmonics

$$\langle \theta\phi|\ell,m\rangle = Y^\ell_m(\theta,\phi)$$
7. [10 pts] Two particles of mass \( M_1 \) and \( M_2 \) are attached to a massless rigid rod of length \( d \). The rod is attached to an axle at its center-of-mass, and is free to rotate without friction in the x-y plane.

Describe the Hilbert space of the system and then write the Hamiltonian. What are the energy levels and degeneracies? What are the wavefunctions of the energy eigenstates?

Only a single angle, \( \phi \) is required to specify the state of the system, where \( \phi \) is the azimuthal angle, thus the Hilbert space is \( \mathcal{H}(\phi) \).

The Hamiltonian is then
\[
H = \frac{L_z^2}{2I}
\]
where
\[
I = 2M \left( \frac{d}{2} \right)^2 = \frac{Md^2}{2}
\]
is the moment of inertia. This gives
\[
H = \frac{L_z^2}{Md^2}
\]
The energy levels are then
\[
E_m = \frac{\hbar^2 m^2}{Md^2}
\]
where \( m = 0, \pm 1, \pm 2, \pm 3, \ldots \)
The energy levels all have a degeneracy of 2, except for \( E_0 \), which is not degenerate.

The wavefunctions are given by
\[
\langle \phi | m \rangle = \frac{e^{im\phi}}{\sqrt{2\pi}}
\]
8. [10 pts] For a two-particle system, the transformation to relative and center-of-mass coordinates is defined by

\[ \vec{R} = \vec{R}_1 - \vec{R}_2 \]
\[ \vec{R}_{CM} = \frac{m_1 \vec{R}_1 + m_2 \vec{R}_2}{m_1 + m_2} \]

The corresponding momenta are defined by

\[ \vec{P} = \mu \frac{d}{dt} \vec{R} \]
\[ \vec{P}_{CM} = M \frac{d}{dt} \vec{R}_{CM} \]

where \( \mu = \frac{m_1 m_2}{M} \) is the reduced mass, and \( M = m_1 + m_2 \) is the total mass. Invert these expressions to write \( \vec{R}_1, \vec{R}_2, \vec{P}_1, \) and \( \vec{P}_2 \) in terms of \( \vec{R}, \vec{R}_{CM}, \vec{P}, \) and \( \vec{P}_{CM} \).

The solutions are

\[ \vec{R}_1 = \vec{R}_{CM} + \frac{m_2}{M} \vec{R} \]
\[ \vec{R}_2 = \vec{R}_{CM} - \frac{m_1}{M} \vec{R} \]

Writing \( \vec{P} \) and \( \vec{P}_{CM} \) in terms of \( \vec{P}_1 \) and \( \vec{P}_2 \) gives

\[ \vec{P} = \frac{m_2 \vec{P}_1 - m_1 \vec{P}_2}{m_1 + m_2} \]
\[ \vec{P}_{CM} = \vec{P}_1 + \vec{P}_2 \]

Inverting this gives

\[ \vec{P}_1 = \frac{m_1}{M} \vec{P}_{CM} + \vec{P} \]
\[ \vec{P}_2 = \frac{m_2}{M} \vec{P}_{CM} - \vec{P} \]